Appearance of a bound state in the two-body problem and the ϵ expansion

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A method is proposed for calculating the critical coupling constant g_c, at which a bound state arises, in the three-dimensional two-body problem. The critical value g_c is first calculated in a space of dimensionality $d = 2 + \epsilon$, as a power series in ϵ, and then the result is extrapolated to $\epsilon = 1$. Even the first two terms of the ϵ expansion give a good approximation of g_c.

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In the three-dimensional two-body problem with a short-range interaction potential, a bound state does not exist at an arbitrary value of the coupling constant g, but only at $g \geqslant g_c$, where g_c is a critical value which depends on the particular potential. The critical coupling constant g_c is an important characteristic of the interaction potential and frequently must be known very accurately. Since it is by no means always possible to solve the Schrödinger equation analytically, approximate and numerical methods become very important.

In this letter we propose a simple method for calculating g_c approximately in the two-body problem (for the s state) for an arbitrary short-range potential. The underlying idea is to work from the circumstance that in two-dimensional space we have $g_c = 0$, since in this case an arbitrarily weak potential will give rise to a bound state. If we then transform to a space of dimensionality $d = 2 + \epsilon$, we can derive[1)] an expression for $g_c(\epsilon)$ as a power series in ϵ. To find g_c we must set $\epsilon = 1$ in each term of the series (cf. the definition of the critical indices in the theory of phase transitions by the ϵ-expansion method[2]). As we will see below, even the first two terms of the expansion of $g_c(\epsilon)$ in the series in ϵ yield a rather good approximation of g_c at $\epsilon = 1$ for a wide variety of potentials.

We begin from the Schrödinger equation for the s-state wave function, written in a space of dimensionality $d = 2 + \epsilon$ ($\hbar = 2m = 1$):

$$-\frac{1}{r^{\epsilon+1}}\frac{\partial}{\partial r}\left(r^{\epsilon+1}\frac{\partial\Psi}{\partial r}\right) - gV(r)\Psi = E\Psi. \quad (1)$$

Positive values of g correspond to an attraction. The substitution $\Psi = \phi r^{-\epsilon/2}$ puts Eq. (1) in the form

$$\frac{d^2\phi}{dr^2} + \frac{1}{r}\frac{d\phi}{dr} + \left[gV(r) + E - \frac{\epsilon^2}{4r^2}\right]\phi = 0. \quad (2)$$

This is simply a two-dimensional radial Schrödinger equation for motion with an angular momentum $m = \epsilon/2$. Restricting the analysis to the case $E \leqslant 0$, we can rewrite Eq. (2) in the integral form

$$\phi(r) = -g\int_0^\infty dr'\, G_\epsilon(r, r'; \kappa)\, V(r')\,\phi(r'), \quad (3)$$

where $\kappa = \sqrt{-E}$, and the Green's function is

$$G_\epsilon(r, r'; \kappa) = -r' I_{\epsilon/2}(\kappa r_<) K_{\epsilon/2}(\kappa r_>),$$

where $r_>$ ($r_<$) is the greater (lesser) of r, r' (the definition of the modified Bessel functions $I_{\epsilon/2}$ and $K_{\epsilon/2}$ corresponds to Ref. 3). Since we are interested in the threshold value g_c, which corresponds to a bound state with a zero energy, we must take the limit $\kappa \to 0$ in Eq. (3). As a result, we find the equation

$$\phi(r) = \frac{g_c(\epsilon)}{\epsilon}\int_0^\infty dr' r' \left(\frac{r_<}{r_>}\right)^{\epsilon/2} V(r')\,\phi(r'), \quad (4)$$

which serves as a definition of $g_c(\epsilon)$. In the limit of small ϵ, this expression takes the particularly simple form

$$\phi(r) = \frac{g_c(\epsilon)}{\epsilon}\int_0^\infty dr' r' V(r')\,\phi(r')$$

and has the obvious solution $\phi(r) = \text{const}$ in this case. We then find immediately that in the limit $\epsilon \to 0$ we have

$$g_c(\epsilon) \cong \epsilon\left[\int_0^\infty dr\, r\, V(r)\right]^{-1}. \quad (5)$$

With $\epsilon = 1$ the value of g_c given by (5) agrees [if $V(r) \geqslant 0$ for all r] with the estimate of g_c which can be found from the Bargmann inequality for the s state:

$$n_0 \leqslant g\int_0^\infty dr\, r\, |V(r)|,$$

where n_0 is the number of bound states with a zero angular momentum.

Working from Eq. (4) we can easily construct a regular perturbation theory in ϵ for $\phi(r)$ and $g_c(\epsilon)$ and thereby refine the estimate based on the Bargmann inequality. When the second term of the ϵ expansion is taken into account, the expression for $g_c(\epsilon)$ becomes

$$g_c(\epsilon) = \frac{\epsilon}{\int_0^\infty dr\, r V(r)} + \frac{\epsilon^2}{2} \frac{\int_0^\infty dr\, r \int_0^\infty dr' r' V(r) V(r') \ln \frac{r_>}{r_<}}{\left[\int_0^\infty dr\, r\, V(r) \right]^3} + \ldots \qquad (6)$$

The integrals in this expression can easily be evaluated for a wide variety of potentials. Table I summarizes the results of corresponding calculations and shows the resulting estimates of the threshold coupling constants (for the four simplest potentials). The last column of this table shows the exact values of g_c (found either through a numerical solution of the Schrödinger equation or by numerical methods).

The good agreement between the results found from expression (6) and the exact values of g_c is remarkable. Expression (6) can thus apparently be regarded as an independent estimate[3] of g_c with $\epsilon = 1$. To evaluate, or improve, its accuracy we would of course need to calculate the next term in the ϵ expansion. However, the general expression for this next term is too lengthy to be reproduced here.

The structure of the power series in ϵ undoubtedly depends on the particular potential,[4] but we believe that certain aspects of this structure can be seen in the example of the exactly solvable (at arbitrary ϵ) problem of a square potential well, $V(r) = \theta(a - r)$. In this case $g_c(\epsilon)$ satisfies the equation

$$J_{\epsilon/2-1}\left(\sqrt{g_c(\epsilon)}\, a\right) = 0,$$

and the ϵ -expansion series is

$$g_c(\epsilon)\, a^2 = 2\epsilon + \frac{1}{2}\epsilon^2 - \frac{1}{24}\epsilon^3 + \frac{7}{576}\epsilon^4 - \frac{293}{69020}\epsilon^5 + \ldots \qquad (7)$$

TABLE I.

$V(r)$	$g_c(\epsilon) a^2$ from (6)	$g_c a^2$ from (6) at $\epsilon = 1$	Exact values $g_c a^2$
$a^4/(r^2 + a^2)^2$	$2\epsilon + \epsilon^2$ [2]	3	3
$\theta(a - r)$	$2\epsilon + \epsilon^2/2$	2.5	2.464...
$\frac{a}{r} e^{-r/a}$	$\epsilon + \epsilon^2 \ln 2$	1.6931...	1.6798...
$e^{-r/a}$	$\epsilon + \epsilon^2(\ln 2 - 1/4)$	1.4431	1.4457 ...

We see that, beginning with the third term, the series becomes an alternating-sign series, and the coefficients in the series fall off rapidly [Eq. (4) suggests that the actual expansion parameter is $\epsilon/2$, rather than simply ϵ itself]. Because of these two circumstances, a good approximation of g_c can be found from simply the first two terms of series (7).

It would be interesting to generalize this method to the case of a three-body system, but attempts in this direction have so far been unsuccessful.

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[1)]There is an analogy here with the calculation[1] of the phase-transition temperature in the nonlinear $O(N)$ σ model with $d = 2 + \epsilon$.

[2)]It can be shown that this is the exact result, valid at all ϵ.

[3)]For the potentials listed in Table I, expression (6) gives better values for g_c than the condition recently derived by Rosen.[4]

[4)]The series may have a finite number of terms (as it does, for example, for the first potential in Table I), but in the general case of an arbitrary potential the series may be asymptotic.

[1]A. M. Polyakov, Phys. Lett. **59B**, 79 (1975).
[2]S. Ma, Critical Phenomena, Benjamin, New York, 1976 (Russ. transl. Mir, Moscow, 1980).
[3]Spravochnik po spetsial'nym funktsiyam (Handbook of Special Functions), Nauka, Moscow, 1979.
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