Spontaneous compaction in Kaluza–Klein models and the Casimir effect

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A particular mechanism for spontaneous compaction in Kaluza–Klein models is analyzed. The compaction in this case results from the quantum vacuum expectation value of the energy-momentum tensor of the matter fields. It is shown that the space $M^4 \times S^d$ is a solution of the effective Einstein equations.

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Models of the Kaluza–Klein type[1] have been discussed extensively in recent years in connection with the effort to unify gravitation and gauge fields[2] and to construct expanded supergravity models.[3] Here it is necessary to compact the original $(4 + d)$-dimensional space in order to make the ground state—the direct product of the non-compact four-dimensional space-time and the compact d-dimensional inner space—a solution of the corresponding Einstein equations. All the compaction mechanisms that have been discussed[4] use nontrivial classical solutions of the Einstein equations with auxiliary matter fields.

We wish to propose another possible compaction mechanism, according to which the energy-momentum tensor on the right side of the Einstein equations arises not from the classical solutions but as a result of incorporating single-loop corrections associated with the compactness of the d-dimensional space. This is none other than the well-known Casimir effect.[5] Appelquist and Chodos[6] have studied the Casimir effect in the case of the compaction of $M^4 \times S^1$ in connection with questions related to the stability of the vacuum.

The Einstein equations in a $(4 + d)$-dimensional space are

$$R^{(d)}_{AB} - \frac{1}{2} g_{AB} R^{(d)} = \kappa (T_{AB} + g_{AB} \Lambda), \tag{1}$$

where κ is the $(d + 4)$-dimensional gravitational constant, which is related to the Newtonian constant κ_N by $\kappa = \kappa_N V(S^d)$; $V(S^d)$ is the volume of S^d; $A, B = 0, 1, \ldots, 4 + d - 1$; T_{AB} is the vacuum expectation value of the energy-momentum tensor; and the value of the Λ term is determined from the vanishing of the curvature of the four-dimensional space M^4, If we wish the solution of (1) to be an $M^4 \times S^d$ space, then we have

$$g_{AB} = \left(\begin{array}{c|c} \eta_{\mu\nu} & \\ \hline & g_{ab}(\theta_i) \end{array} \right); \qquad R_{AB} = \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & \frac{R}{d} g_{ab} \end{array} \right), \tag{2}$$

where R is the scalar curvature of S^d, $R = d(d - 1)/r^2$; r is the radius of the S^d sphere;

and θ_i are the coordinates on the sphere. Substituting (2) into the left side of (1), we find

$$\kappa(T_{AB} + g_{AB}\Lambda) = \left(\begin{array}{c|c} -\dfrac{d(d-1)}{2r^2}\eta_{\mu\upsilon} & 0 \\ \hline 0 & -\dfrac{(d-1)(d-2)}{2r^2}g_{ab} \end{array}\right). \tag{3}$$

If (3) is to have a solution, the conditions $\langle 0|T_{\mu\upsilon}|0\rangle = T_1\eta_{\mu\upsilon}$, $T_{a\mu} = 0$, $\langle 0|T_{ab}|0\rangle = T_2 g_{ab}$, must hold. These conditions naturally must hold for the vacuum expectation value of the energy-momentum tensor in view of the symmetry of the spaces M^4 and S^d. From (3) we find

$$\kappa(T_1 + \Lambda) = -\frac{d(d-1)}{2r^2}\ ; \quad \kappa(T_2 + \Lambda) = -\frac{(d-1)(d-2)}{2r^2}\ . \tag{4}$$

For a conformally invariant coupling of matter with gravity we have the relation $4T_1 + dT_2 = 0$, which is related to the trace-free nature of T_{AB}. This assertion is valid if there are no anomalies[1] in T^A_A (Ref. 7). In general, we would have $g^{AB}(x)\langle 0|T_{AB}(x)|0\rangle = A(x)$, where $A(x)$ is expressed in terms of R^2_{ABCD} and R^2_{AB}. It is important to note that for the maximally symmetric spaces, including S^d, these quantities are actually independent of x, so that $\langle 0|T^B_B|0\rangle = A$ is a constant, determined exclusively by the sphere radius r and the nature of the matter fields. For simplicity, we set $A = 0$ at this point, although all the results can easily be generalized to the case $A \neq 0$ and will be discussed in a separate detailed paper.

Since $T_1 = T_{00}$ is the energy density in the $d+4$ space, it is given by

$$T_1 = \frac{1}{2V(S^d)}\left[\sum_i \int \frac{d^3k}{(2\pi)^3} N_i \sqrt{\mathbf{k}^2 + \frac{m_i^2}{r^2}}\right]_{\text{reg}}, \tag{5}$$

where m_i^2 are the eigenvalues of the operator of the corresponding equation of motion of the matter field, which is bounded on the compact subspace S^d, and N_i is the number of states with the given m_i. A different expression for T_1 was used in Ref. 6:

$$T_1 = -\frac{i}{2V(S^d)}\left[\sum_i \int \frac{d^4k}{(2\pi)^4} N_i \ln\left(k^2 + \frac{m_i^2}{r^2}\right)\right]_{\text{reg}}, \quad k^2 = \mathbf{k}^2 - k_0^2\ .$$

It is a simple matter to show that these two representations lead to the same result for the regularized $\langle 0|T_{AB}|0\rangle$. A calculation of the Casimir effect does not depend[8] on the choice of a regularization method for (5), and it leads to an expression of the type $T_1 = C/r^{4+d}$. The constant C can be calculated unambiguously; it depends on the particular choice of matter fields. From (4) we then find expressions for that value of Λ term, which provides the required compaction of $M^4 \times S^d$ and for the compaction radius r_0:

$$r_0^{2+d} = -\kappa C \frac{(d+4)}{d(d-1)}; \qquad \Lambda = -\frac{d(d-1)(d+2)}{2r_0^2(d+4)}. \tag{6}$$

It follows from (6) that for our purposes we must have $C<0$.

Up to this point, we have been talking about the energy-momentum tensor of matter fields. Appelquist and Chodos[6] have calculated an effective potential (i.e., the component T_{00}) which results from quantum fluctuations in the gravitational field above a given background metric. It may be that a tensor structure of the type in (3) arises in this case; this circumstance would mean a purely dynamic compaction of the space in the absence of additional matter fields. However, the validity of this suggestion must be looked at more carefully (R^2 are the contraterms in a single loop).

This compaction mechanism does not work for the $M^4 \times S^1$ model because of the zero curvature of M^4 and S^1. The first nontrivial dimensionality is $(4+2)$.

Let us briefly[2)] discuss the calculation of the Casimir effect for the model of a compaction of $M^4 \times S^2$, in which the matter fields are a massless scalar field and a massless spinor field. The boson spectrum in this case is well known:

$$m_i^2 = i(i+1); \qquad N_i = 2i+1, \qquad i = 0, 1, \ldots, \tag{7}$$

where m_i and N_i are the same as in (5).

The problem of calculating the fermion spectrum is less trivial, because of the spin-gravity interaction. The spectrum found by us has three branches:

$$\begin{aligned} &m_f^2 = \left(f+\frac{1}{2}\right)^2; && N_f = 2f, && f = 2, 3, \ldots \\ &m_f^2 = f^2; && N_f = 2, && f = 1, 2, \ldots \\ &m_f^2 = (2f)^2; && N_f = 2, && f = 1, 2, \ldots \\ &m_f^2 = 0; && N_f = 1. && \end{aligned} \tag{8}$$

We note that (7) and (8) each have a single boson and a single fermion zero mode; their presence means that the supersymmetry is conserved only for the zero modes upon a spontaneous compaction of the supersymmetry theory; the supersymmetry is broken for the mass states. This residual supersymmetry does not depend on the sphere radius r, in agreement with the generally accepted reduction technique in supergravity.[3,4]

Knowing (7) and (8), and using the standard regularization methods,[8] we can evaluate (5) for a specific model. Here we must take into account the circumstances that the contribution of the fermions to the energy of the vacuum enters with a minus sign. We finally find

$$C_b = -1.1 \times 10^{-4}; \qquad C_f = 8.6 \times 10^{-4}. \tag{9}$$

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After this paper had been submitted for publication, we learned of the preprint[9] by Candelas and Weinberg, who discussed some corresponding questions. We wish to thank D. Gross and V. I. Zakharov for bringing this preprint to our attention.

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[2)]The detailed calculations will be published elsewhere.

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