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Commutativity equations and dressing transformations

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We study dressing transformations that generate all solutions to Commutativity equations, and, after picking up special coordinates, all solutions to WDVV equations. We conjecture that the homological tensor product of solutions to Commutativity equations corresponds to the tensor product of matrixes of dressing transformation and check it in the first nontrivial case.

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The main problem in our current understanding of the theory of quantum gravity is the uniqueness of the M-theory. In order to understand it we need to find and to study much simpler and tractable models that have common features with M-theory. That is why so much attention is spend on so called topological strings [1 – 12]. Moreover, recent studies of superstrings made by Berkovits [13] indicate that probably even superstrings themselves could be considered somehow as a rather special case of topological strings.

From the very first days of topological strings it was found that their tree level amplitudes satisfy remarkable quadratic equations (WDVV or Associativity equations). Moreover, it turns out that in many known examples higher loop amplitudes could be expressed in terms of the tree level amplitudes.

All this leads to the following scheme of study of topological theories. First, we classify solutions to WDVV equations. Second, we look for conditions on solutions to WDVV equations that correspond to topological strings.

In this paper we give a classification of all solutions to WDVV equations based on their relations to solutions to Commutativity equations [14, 15, 16, 17] on $GL(\dim W)$-valued functions on the base space S.

In particular we will classify all solutions to Commutativity equations in terms of the maximal commutative subgroups of $GL(\dim W)$ and dressing transformations matrixes that are matrixes whose elements are polynomials of one variable.

Moreover, the parameterization of solutions to Commutativity equations in terms of dressing matrixes looks rather natural when we study the tensor product of solutions in this parametrization.

Really, recall that solutions to Commutativity equations can be obtained from supersymmetric quantum mechanics. Therefore, one can define the tensor product by considering tensor product of quantum mechanics – the total Hilbert space is a product of Hilbert spaces, while total supercharges are sums of supercharges. As it was shown in [16] this tensor product equals to the homological tensor product on solutions to Commutativity equations. Really, each solution corresponds to a factorizable map to cohomologies of the moduli space $\bar{L}_n$, and a tensor product can be obtained by taking product in cohomologies of the moduli spaces.

In this paper we conjecture that the abovementioned tensor product corresponds just to tensor product of maximal commuting subalgebras and to tensor product of dressing transformation matrixes. We check this conjecture in the first nontrivial case and find that it really works.

In [18] we studied procedure of reconstruction of solutions to WDVV equations from solutions to Commutativity equation and found that homological tensor product is compatible with the reconstruction procedure. Therefore, we manage not only to classify solutions to WDVV but also we found parametrization in which tensor product on such solutions takes a rather simple form.

All this implies that the theory of WDVV equation and the theory of its quantization in the spirit of [19, 20] should be rewritten in terms of dressing matrix. We should mention that similar construction for the semisimple case was given by [21], but from our presentation it should be clear that the semisimplicity condition is an auxiliary assumption.

Commutativity equations [14, 17 – 19] are the set of equations on $\tau(t)$ – a $GL(\dim W)$-valued formal series in $t_1, \dots, t_n$ (later, we would like to treat $t_1, \dots, t_n$ as a set of coordinates on a n-dimensional space S.

If we choose a basis e_a in the vector space W, then we can consider $\tau(t)$ as a matrix $\tau_a^b(t)$ taking values in the formal series in $t_1, \dots, t_n$.

The Commutativity equations have the form:

$$d\tau \wedge d\tau = 0 \tag{1}$$

or, in components,

$$\frac{\partial\tau_a^b(t)}{\partial t_i}\frac{\partial\tau_b^c(t)}{\partial t_j} = \frac{\partial\tau_a^b(t)}{\partial t_j}\frac{\partial\tau_b^c(t)}{\partial t_i}. \tag{2}$$

We will use the following properties of Commutativity equations:

A) Suppose that we have a map $f : S' \subset S$,

$$t_i = f_i(t'). \tag{3}$$

Then $f^*\tau$ is an induced solution to Commutativity equations, where

$$(f^*\tau)(t') = \tau(f_1(t'), ..., f_n(t')). \tag{4}$$

B) Commutativity equations are equivalent to the flat connections in the trivial bundle W over S with the spectral parameter z, namely, if

$$\nabla(z) = d + z^{-1}A \tag{5}$$

then from

$$(\nabla(z))^2 = 0, \tag{6}$$

it follows that (from the terms linear in z^{-1})

$$A = d\tau \tag{7}$$

and that (from the terms quadratic in z^{-1})

$$A^2 = 0 \tag{8}$$

that is Commutativity equations.

Now we begin our description of the classification of solutions to Commutativity equations.

Due to property A) it is reasonable to consider only the primitive solution, i.e. solutions with $\dim W = \dim S$ and such that these solutions could not be induced from the solution with $\dim S < \dim W$.

In the classification of primitive solutions we use the dressing technique (we are grateful to V.Fock for discussion on this subject) that works as follows:

We start with the maximal commutative subalgebra in End(W), that has $\dim W$ generators ϕ_i. It corresponds to the simple solution of Commutativity equations

$$\tau = \phi = \sum_{i=1}^{\dim} \phi_i t_i. \tag{9}$$

This corresponds to connection

$$d + z^{-1}\sum_i \phi_i dt_i = \exp(-z^{-1}\phi) d \exp(z^{-1}\phi). \tag{10}$$

Consider arbitrary matrix

$$U(z) = \exp(\sum_{k=1}^{\infty} z^k V_k) \tag{11}$$

that is holomorphic function of z, and consider factorization problem

$$\exp(z^{-1}\phi(t))U(z) = M(t,z)N(t,z^{-1}), \tag{12}$$

such that $N(t,0) = 1$ Then,

$$N(t,z^{-1}) = M(t,z)^{-1}\exp(z^{-1}\phi(t))U(z), \tag{13}$$

thus

$$N(t,z^{-1})dN^{-1}(t,z^{-1}) = M(t,z)^{-1}(d + z^{-1}d\phi)M(t,z). \tag{14}$$

The left hand side of the above equation has the form $d + z^{-1}A_1(z^{-1},t)$ while the right hand side is obviously $d + z^{-1}A_2(z,t)$. Thus,

$$A_1(z^{-1},t) = A_2(z,t) = A(t) \tag{15}$$

and (from the expression for A_2) it follows that

$$A(t) = M(0,t)^{-1}\phi_i M(0,t) dt_i \tag{16}$$

and that $A(t) = d\tau$, and that this τ solves Commutativity equations.

In order to get explicit expressions for $M(0,t)$ – consider the level zero representation of the $GL(\dim W)$ current algebra. Consider the vacuum $|a>$ and covacuum $<b|$ that are annihilated by positive and negative modes of currents and that form fundamental representations for the zero modes of currents. Then, $M(0,t)$ is a tau-function:

$$M(0,t)_a^b = <b|\widehat{\exp(z^{-1}\phi)}\widehat{U(z)}|a>, \tag{17}$$

where $\widehat{K(z,z^{-1})}$ denotes the operator corresponding to the matrix K in the level zero representation.

Below we will present some explicit formulas for first terms in expansion of τ in terms of dressing transformation parameters V.

The log of the dressing matrix $M(0,t)$ up to the third order in t is equal to:

$$\log M_0(t) = V_1' + \frac{1}{2}V_2'' + \frac{1}{2!2!}[V_1'', V_1] + \\ + \frac{1}{3!2!2!}[V_1'', [V_1', V_1]] + \frac{1}{3!2!2!}[[V_1'', V_1'], V_1] + \\ + \frac{1}{3!3!}[[V_1''', V_1], V_1] + \frac{1}{3!2!}[V_2''', V_1] + \\ + \frac{1}{3!2!}[V_1''', V_2] + \frac{1}{2!2!}[V_1'', V_2]] + \frac{1}{3!}V_3'''' + O(t^4). \tag{18}$$

The expression for τ up to the fourth order in t is as follows:

$$\tau(t) = \phi + \frac{1}{2!}V_1'' + \frac{1}{3!2!}[V_1''', V_1] + \frac{1}{3!}V_2''' + \\ + \frac{1}{2!2!}[V_1'', V_1'] + \frac{1}{3!2!2!}[V_1'', [V_1'', V_1]] + \\ + \frac{1}{3!3!}[[V_1''', V_1'], V_1] + \frac{1}{3!3!}[[V_1''', V_1], V_1'] + \\ + \frac{1}{4!3!}[[V_1'''', V_1], V_1] + \frac{1}{3!2!}[[V_1'', V_1'], V_1'] + \\ + \frac{1}{4!2!}[V_2'''', V_1] + \frac{1}{4!2!}[V_1'''', V_2] + \frac{1}{3!2!}[V_2''', V_1'] + \\ + \frac{1}{3!2!}[V_1''', V_2'] + \frac{1}{2!2!2!}[V_1'', V_2''] + \frac{1}{4!}V_3'''' + O(t^5), \quad (19)$$

where $,V_i' = [\phi, V_i]$, $V_i'' = [\phi, [\phi, V_i]]$, $V''' = [\phi, [\phi, [\phi, V_i]]]$...etc.

Explicit formulas for solutions to Commutativity equations suggest that one could obtain a universal formula as a seria in the Lie algebra generated by matrixes V_k and ϕ_i. Conjecturally, the terms in this seria are classified by 3-valent rooted trees, combinatorial coefficient is something like inverse factorial for the number of ϕ times inverse factorial for the number of V's. It suggests that Commutativity equation could be proved directly through graph reasoning without appealing to dressing transformations. We leave this problem for future work.

One can show that the general solution can be induced from the solution obtained by dressing transformation.

After classification of solutions to Commutativity equations we describe the procedure of reconstruction of solutions to Associativity (WDVV) equations.

We begin with the acyclic Associativity equations.

The Associativity equations are not the only equations that one can associate with the Deligne-Mumford compactification of the moduli space of marked points on the sphere. The reason for this is that one can mark all points as "in" points or "out" points. Consider a subspace of markings that contains one "out" point and all other points are marked as "in" points. When the sphere degenerates into two spheres it produces two points – one "in" point (on the component that contains the "out" point) and one "out" point (on the component that contained only "in" points). If we associate vector spaces to "in" points and dual vector spaces to "out" points we can postulate that degeneration is accompanied with the canonical pairing between vector space associated to the "in" point and its dual associated to the "out" point. Reasoning like in the standard derivation of Associativity equations from the Keel relations we see that now the generating function for the correlators is a vector field $v(T)$ on the base space W, equipped with the special coordinates T, such that its second derivatives form structure constants of associative algebra – this is what we will call acyclic Associativity equation:

$$\frac{\partial^2 v^e(T)}{\partial T^a \partial T^b}\frac{\partial^2 v^d(T)}{\partial T^e \partial T^c} = \frac{\partial^2 v^e(T)}{\partial T^a \partial T^c}\frac{\partial^2 v^d(T)}{\partial T^e \partial T^b}. \quad (20)$$

Solutions to acyclic Associativity equations could be obtained from the solutions to Commutativity equations if we suppose that the latter have a primitive element – a vector $h \in W$, such that the operator $d\tau(h)$ considered as an operator from the tangent space to the base S to the space W is nondegenerate.

The construction goes as follows (see [15, 18]). Consider a map from S to W, that sends point on V with coordinates t to the point on W with coordinates $T^b(t)$ given by:

$$T^b(t) = \tau_a^b(t)h^a. \quad (21)$$

Consider the inverse map f from W to S , it expresses t^i as function of T^a:

$$t^i = f^i(T; h); T^b = \tau_a^b(f(T, h))h^a. \quad (22)$$

Then one can show that there exists a vector field $v^a(T)$, such that

$$f^*(\tau)_a^b = \partial v^b/\partial T^a \quad (23)$$

and this vector field solves acyclic Associativity equations.

The special feature of this choice of coordinates is that coordinate in the direction of h enters v only linearly, namely:

$$h^a \frac{\partial^2 v^b}{\partial T^a \partial T^c} = \delta_b^c. \quad (24)$$

There are other choices of coordinates that still produce acyclic Associativity equations, they are related to lifting to the action gravitational descendants and they will violate property (24). We will discuss this topic elsewhere.

By Associativity equations (without Euler vector field) we mean the following equations on a function $F(T)$ on the space W:

$$\frac{\partial^3 F(T)}{\partial T^a \partial T^b \partial T^e}\eta^{ef}\frac{\partial^3 F(T)}{\partial T^f \partial T^c \partial T^d} = \\ = \frac{\partial^3 F(T)}{\partial T^a \partial T^c \partial T^e}\eta^{ef}\frac{\partial^3 F(T)}{\partial T^f \partial T^d \partial T^d} \quad (25)$$

here η is a constant pairing on the space W.

Suppose that we have solution to acyclic Associativity equations, and there is a constant metric η, such that

$$dT^b v^a \eta_{ab} = dF(T) \qquad (26)$$

then it is easy to show that $F(T)$ is a solution to Associativity equations. Moreover, if we introduce a special coordinate

$$T_0 = h^a T^b \eta_{ab} \qquad (27)$$

then one can show that

$$\frac{\partial^3 F(T)}{\partial T_0 \partial T_a \partial T_b} = \eta_{ab} \qquad (28)$$

i.e. we get solutions to WDVV with the identity.

We call $\tau(t)$ a symmetric (with respect to η) solution to commutativity equations, if $\tau^T = \tau$, i.e.

$$\tau_a^b \eta_{bc} = \tau_c^b \eta_{ba}. \qquad (29)$$

Suppose that symmetric solution to Commutativity equations admits primitive element h. Then solution to acyclic Associativity equation will satisfy the property (26), i.e. will lead to solution to Associativity equations with identity.

Thus, in order to get solutions to Associativity equations we need to study symmetric solutions to Commutativity equations. As one can show (and check using the manifest formula), we need impose the following conditions on ϕ and V_k:

$$\phi^T = \phi, V_k^T = (-1)^{k+1} V_k \qquad (30)$$

where symmetry is studied with respect to metric η.

We have already seen how useful dressing transformation is in studying Commutativity equations and their ability to be promoted to WDVV equations. Now we will show that dressing transformation parametrization probably drastically simplifies the computation of the homological tensor product of solutions on Commutativity equations. Namely, we will compare two tensor products. In order to write down explicit formulas we need to introduce coefficients of the expansion of solution to Commutativity equations in parameters t:

$$\tau(t) = \sum_n \sum_{i_1 \ldots i_n} \tau_{i_1 \ldots i_n} \frac{t_{i_1} \ldots t_{i_n}}{n!}. \qquad (31)$$

First tensor product on solutions to Commutativity equations comes from the tensor products on commutative algebras and on matrixes of dressing transformations. Namely, if e_i is a basis in S_1, and $e_{i'}$ is a basis in S_2, then we take

$$e_I = e_{ii'} = e_i \otimes e_{i'} \qquad (32)$$

to be a basis in $V^{1 \otimes 2} = V^1 \otimes \tilde{V}^2$. Thus, we have

$$\phi_I^{1 \otimes 2} = \phi_i \otimes \phi_{i'} \qquad (33)$$

and

$$V_m^{1 \otimes 2} = 1 \otimes \tilde{V}_m + V_m \otimes 1 \qquad (34)$$

The latter formula can be also rewritten in a suggestive form for the matrixes of dressing transformations

$$U^{1 \otimes 2}(z) = U(z) \otimes \tilde{U}(z), \qquad (35)$$

where U and $\tilde{U}$ are the matrix of dressing transformations for the first and the second solutions to Commutativity equations.

The explicit formulas for the tensor product up to the third order (formulas for the first two orders are rather simple), look as follows:

$$\tau_I^{1 \otimes 2} = \tau_i \otimes \tau_{i'}, \qquad (36)$$

$$\tau_{IJ}^{1 \otimes 2} = \tau_i \tau_j \otimes \tau_{i'j'} + \tau_{ij} \otimes \tau_{i'} \tau_{j'}, \qquad (37)$$

$$\begin{aligned} \tau_{IJK}^{1 \otimes 2} &= \tau_{ijk} \otimes \phi_{i'} \phi_{j'} \phi_{k'} + \phi_i \phi_j \phi_k \otimes \tau_{i'j'k'} + \\ &+ \frac{1}{2}(\tau_{ij} \phi_k \otimes \phi_{i'} \phi_{j'} [\phi_{k'}, \tilde{V}_1] - \\ &- \phi_k \tau_{ij} \otimes [\phi_{k'}, \tilde{V}_1] \phi_{i'} \phi_{j'} + \\ &+ \tau_{ik} \phi_j \otimes \phi_{i'} \phi_{k'} [\phi_{j'}, \tilde{V}_1] - \phi_j \tau_{ik} \otimes [\phi_{j'}, \tilde{V}_1] \phi_{i'} \phi_{k'} + \\ &+ \phi_i \tau_{jk} \otimes \phi_{i'} [\phi_{j'} \phi_{k'}, \tilde{V}_1] - \tau_{jk} \phi_i \otimes [\phi_{j'} \phi_{k'}, \tilde{V}_1] \phi_{i'} + \\ &+ \phi_i \phi_j [\phi_k, V_1] \otimes \tau_{i'j'} \phi_{k'} - [\phi_k, V_1] \phi_i \phi_j \otimes \phi_{k'} \tau_{i'j'} + \\ &+ \phi_i \phi_k [\phi_j, V_1] \otimes \tau_{i'k'} \phi_{j'} - [\phi_j, V_1] \phi_i \phi_k \otimes \phi_{j'} \tau_{i'k'} + \\ &+ \phi_i [\phi_j \phi_k, V_1] \otimes \tau_{j'k'} - [\phi_j \phi_k, V_1] \otimes \phi_{i'} \tau_{j'k'} \phi_{i'}), \end{aligned} \qquad (38)$$

where $\tau_{ij} = [\phi_i, [\phi_j, V_1]]$

The second product comes from the interpretation of the solutions to Commutativity equations as factorizable maps into cohomologies of the moduli space $\bar{L}_n$ introduced in [12, 16, 17, 18].

Recall, that $\bar{L}_n$ is the compactification of $C^{*,n}/C^*$, i.e. the moduli space of n points on the C^* acted by a multiplication on nonzero complex number.

Namely, given a solution to Commutativity equation τ one can construct an element h_n :

$$h_n \in H^*(\bar{L}_n) \otimes S *^{\otimes n} \otimes End(W) \qquad (39)$$

and

$$h_{n,1 \otimes 2} = h_{n,1} h_{n,2} \qquad (40)$$

where the product in the r.h.s of (40) is the product in cohomologies and a tensor product in $S *^{\otimes n} \otimes End(W)$ (note, that $S_{1\otimes 2} = S_1 \otimes S_2$, and $W_{1\otimes 2} = W_1 \otimes W_2$).

In the explicit computation of the product we use that

$$h_{n,1} \cap h_{n,2}(\bar{L}_n) = \sum_{A,B} h_{n,1}(C_A) h_{n,2}(C_B) N^{AB} \quad (41)$$

where C_A stands for the basis of cycles in homologies of $\bar{L}_n$, and N is the inverse to their intersection matrix.

Equivalence of tensor products computed on $\bar{L}_1$ and $\bar{L}_2$ is obvious and , thus, it is not quite representative. Here we will compute and compare two tensor products on $\bar{L}_3$ (i.e. for the term that is of third order in t in expansion of $\tau(t)$).

In particular, (see [18] for details) we obtained the following result for $\bar{L}_3$.

In this case the intersection matrix of 2-cycles has rank 4. Let $(x, y)(z)$ correspond to the cycle where sphere is degenerated into two spheres with two points with labels x, y on the first sphere and point with label z – on the second.

Let j, k, l denote the labels of the marked points.

In the basis of cycles:

$$(jl)(k) + (j)(kl) + (jk)(l); (jk)(l); (jl)(k); (kl)(j) \quad (42)$$

the intersection matrix is diagonal: diag$(1; -1; -1; -1)$. Therefore,

$$\begin{aligned} \tau_{IJK}^{Hom 1\otimes 2} = {} & \tau_{jkl} \otimes \phi_{j'}\phi_{k'}\phi_{l'} + \phi_j\phi_k\phi_l \otimes \tau_{j'k'l'} + \\ & + \tau_{kl}\phi_j \otimes \tau_{k'l'}\phi_{j'} + \tau_{kl}\phi_j \otimes \phi_{l'}\tau_{j'k'} + \\ & + \tau_{kl}\phi_j \otimes \phi_{k'}\tau_{j'l'} + \phi_k\tau_{jl} \otimes \tau_{k'l'}\phi_{j'} + \\ & + \phi_k\tau_{jl} \otimes \phi_{l'}\tau_{j'k'} + \phi_l\tau_{jk} \otimes \tau_{k'l'}\phi_{j'} + \\ & + \phi_l\tau_{jk} \otimes \phi_{k'}\tau_{j'l'} - \phi_j\tau_{kl} \otimes \phi_{j'}\tau_{k'l'}. \end{aligned} \quad (43)$$

By comparing formulas (38) and (43) one can show that they define the same tensor product structure on $\bar{L}_3$.

Thus we have seen that the dressing matrix parametrization of solutions to Commutativity equations is quite effective. The next question is to find representation for the higher loop amplitudes [19, 20] in terms of dressing matrixes.

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