Relaxation of initial perturbation and Landau damping in an inhomogeneous plasma

S. M. Dikman

Institute of Physics Problems, USSR Academy of Sciences
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The relaxation of plasma oscillations in a weakly inhomogeneous anisotropic collisionless plasma is considered. The damping of the Langmuir waves is investigated in detail. The Landau-damping decrement increases exponentially with time, owing to the linear growth of the characteristic wave vector with time.

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The present article is devoted to a solution of the initial problem of relaxation of a perturbation in a one-dimensionally (along the X axis) inhomogeneous plasma. The characteristic wavelengths of the oscillations produced in the plasma are assumed to be small in comparison with the inhomogeneity dimension. It is also assumed that the longitudinal dielectric constant of the plasma $\epsilon(x,\omega,\mathbf{k})=k^{-2}\epsilon_{ij}(x,\omega,\mathbf{k})k_i k_j$ is a monotonic function of the coordinate x. The spectrum of the natural longitudinal oscillations is determined under these conditions by the solutions $\omega(x,\mathbf{k})$ of the dispersion equation

$$\epsilon(x,\omega,\mathbf{k})=0, \qquad (|\operatorname{Im}\omega| << |\operatorname{Re}\omega|). \tag{1}$$

The method used by us makes it possible, in principle, to solve the problem of plasma relaxation independently of the type of the produced natural oscilations. We, however, confine ourselves to a determination of the relaxation time $\tau(x)$ of the plasma waves whose frequency satisfies the inequality

$$\operatorname{Re}\omega(x,\mathbf{k}) >> |k\partial\omega(x,\mathbf{k})/\partial k|. \tag{2}$$

The initial equation for the potential of the electric field ($\omega \ll kc$) is

$$\frac{\partial}{\partial x_i}\left[\int \hat{\epsilon}_{ij}(x,t-t',\mathbf{r}-\mathbf{r}')\frac{\partial\phi(\mathbf{r}',t')}{\partial x_j'}d^3\mathbf{r}'\right] = -4\pi\rho(\mathbf{r},t). \tag{3}$$

We assume satisfaction of the inequalities $k_x a \gg 1$, and $a \gg r_D$ [$a \sim \epsilon/(\partial\epsilon/\partial x)$, r_D is the Debye radius and k_x is the wave-vector component along the X axis]. The kernel in the integral equation (3) is connected with the dielectric tensor by the known relation[1]

$$\epsilon_{ij}(x,k,\omega) - \frac{i}{2}\frac{\partial^2\epsilon_{ij}(x,k,\omega)}{\partial x\partial k_x}$$
$$= \int\int_0^\infty \hat{\epsilon}_{ij}(x,t_0,r_0)\exp(i\omega t_0 - ikr_0)dt_0\, d^3r_0 .$$

Let $x=x_0(\omega,\mathbf{k})$ be a root of Eq. (1). The natural oscillations with given ω and $\mathbf{k}$ are localized near $x_0(\omega,\mathbf{k})$[1]. We now use the weak-inhomogeneity approximation

$$\epsilon_{ij} = a_{ij}(\omega, \mathbf{k}) - \beta_{ij}(\omega, \mathbf{k})(x - x_0(\omega, \mathbf{k})). \tag{4}$$

Here $\alpha_{ij}=\epsilon_{ij}|_{x=x_0}$, $\beta_{ij}=-\partial\epsilon_{ij}/\partial x|_{x=x_0}$, $\alpha_{ij}k_ik_j=0$, $\mathrm{Re}\beta_{ij}k_ik_jk^{-2}=a^{-1}>0$. Taking the inverse Fourier transformation of (4) with respect to ω and k, we obtain the tensor $\hat{\epsilon}_{ij}(x,t_0,\mathbf{r}_0)$, which we substitute in (3). We take next the Fourier transform of (3) and obtain the following equation between the components $\bar{\rho}(\omega,\mathbf{k})$ and $\bar{\phi}(\omega,\mathbf{k})$ (cf.[2])

$$i\beta_{ij}k_ik_j\frac{\partial\bar{\phi}}{\partial k_x} + \left(i\beta_{ix}k_i + \frac{i}{2}\frac{\partial\beta_{ij}}{\partial k_x}k_ik_j - \beta_{ij}k_ik_jx_0\right)\bar{\phi} = -4\pi\bar{\rho}.$$

The solution of this equation is

$$\bar{\phi}(\omega, k) = -\frac{4\pi i}{(\beta_{ij}k_ik_j)^{1/2}}\int_{k_x}^{+\infty}\frac{\bar{\rho}(\mathbf{k}',\omega)}{(\beta_{ij}(\mathbf{k}')k_i'k_j')^{1/2}}$$

$$\times\exp\left[\frac{1}{2}\int_{k_x'}^{k_x}\frac{(\beta_{xi}(\mathbf{k}'')-\beta_{ix}(\mathbf{k}''))}{\beta_{ij}(\mathbf{k}'')k_i''k_j''}dk_x'' - i\int_{k_x'}^{k_x}x_0(\omega, \mathbf{k}'')\,dk_x''\right]dk_x', \tag{5}$$

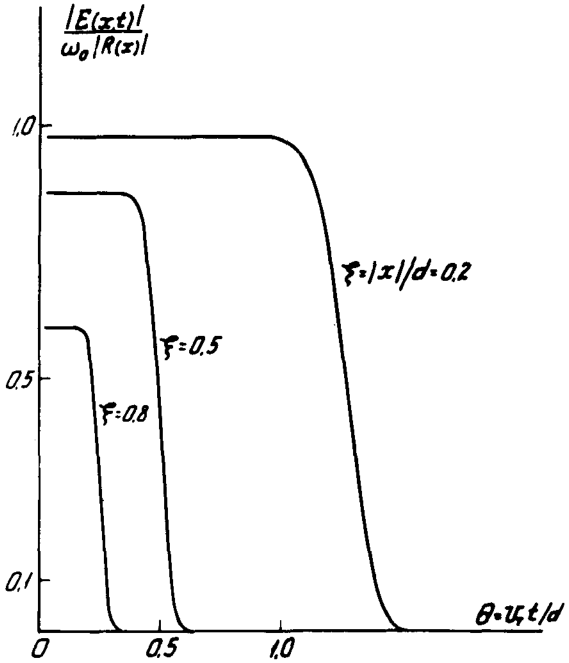
where $\mathbf{k}'$ and $\mathbf{k}''$ are vectors with components $(k'_x,\mathbf{k}_\perp)$ and $(k''_x,\mathbf{k}_\perp)$ respectively. The integration constant is chosen to make the Fourier component of the potential regular as $k_x\to\pm\infty$, and we assume here that the plasma is thermodynamically stable: $\mathrm{Im}\epsilon(\omega,\mathbf{k})>0$. The potential is determined by substituting (5) in the formula

$$\phi(x,t,\mathbf{k}_\perp) = (2\pi)^{-2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\bar{\phi}(\omega, \mathbf{k})\exp(-i\omega t + ik_xx)\,d\omega\,dk_x. \tag{6}$$

We confine ourselves to the case of a sufficiently large characteristic spatial dimension of the imhomogeneity of the extraneous charges, $a\partial\rho/\partial x\lesssim\rho$. Then, taking the condition (2) into account, as well as $k_xa\gg1$, we can take the exponential of (5) outside the integral sign, and put at the same time $k'_x=0$. Assuming the times to be long enough to make $t\partial\rho/\partial t\gg\rho$, we estimate the integral (6) in the geometrical optics approximation. Differentiating the rapidly varying functions in the argument of the exponential with respect to ω and k_x, we obtain the stationary-phase points $k_x=\tilde{k}(x,t,\mathbf{k})$, $\omega=\tilde{\omega}(x,t,\mathbf{k}_\perp)$ from the equations

$$\frac{\partial}{\partial\omega}\int_0^{k_x}x_0(\omega, \mathbf{k})\,dk_x = -t, \qquad x = x_0(\omega, \mathbf{k}). \tag{7}$$

According to (2) and (7), in the zeroth approximation in the parameter $\tilde{k}(\partial\ln\tilde{\omega}/\partial k_x)$ (i.e., neglecting completely the spatial dispersion), $\tilde{k}$ is determined by



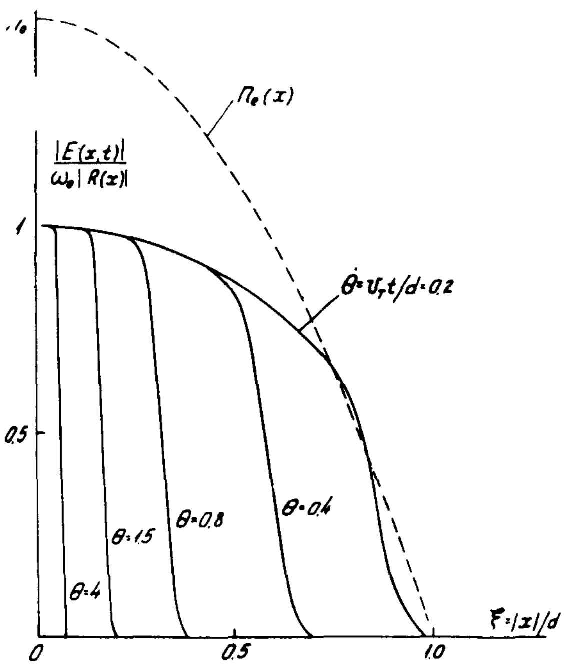


FIG. 1 and FIG. 2. The figures show the relaxation of Langmuir oscillations in a plasma with bell-shaped distribution of the electron density at the value of the parameter $\beta=(\pi/8)^{1/2}a\omega_0/v_{Te}=5\times10^3$.

the expression $\tilde{k} = -t\partial\tilde{\omega}/\partial x$. The linear growth of the wave vector with time arises in our case, as in all problems with a continuous oscillation spectrum, as a result of the "dispersal" of the phases (see[3], as well as the review[4] and the papers,[5–8] in which waves in an inhomogeneous plasma are considered without allowance for kinetic thermal effects). The potential (6) can be represented in the form $\phi(x,t,\mathbf{k}_\perp) = f(x,t,\mathbf{k}_\perp)\exp(i\Psi)$, where the eikonal is equal to

$$i\Psi(x,t,\mathbf{k}_\perp) = -i\tilde{\omega}t + i\tilde{k}x - i\int_0^{\tilde{k}} x_0(\tilde{\omega}, \mathbf{k}')dk'_x = -i\int_0^t \tilde{\omega}(x, t')\,dt'.$$

The characteristic relaxation time should, obviously, be determined from the relation

$$\int_0^\tau \mathrm{Im}\,\tilde{\omega}(x, t')\,dt' \sim -1. \tag{8}$$

We examine in greater detail the Langmuir oscillations in a plasma with inhomogeneity density. From (7) we get

$$\tilde{k} = k_0\left[1 - \frac{3}{2} r^2(t^2\omega_e'^2 + k_\perp^2)\right], \text{ where } k_0 = -t\omega_e'$$

$$\mathrm{Im}\,\tilde{\omega} = \gamma(\tilde{k}(x,t,\mathbf{k}_\perp), \mathbf{k}_\perp) = -\left(\frac{\pi}{8}\right)^{1/2}\frac{\omega_e(x)}{(k_0^2 + k_\perp^2)^{3/2} r^3}\exp\left[-3 + \frac{3}{2}\frac{k_\perp^2}{k_0^2 + k_\perp^2} - \frac{1}{2(k_0^2 + k_\perp^2)r^2}\right].$$

Here $\omega_e^2 = 4\pi n_e(x)e^2/m_e$, $r(x) = v_{Te}/\omega_e$, $\omega'_e = d\omega_e/dx$. Calculating the integral (8) [see formula (11) below] we obtain

$$\tau(x) = \omega_e'^{-1}\left[\frac{1}{2r^2\ln(\omega_e/r|\omega_e'|)} - k_\perp^2\right]^{1/2}$$

This formula is valid if $k_\perp \lesssim \tau\omega'_e \ll r^{-1} \gg \omega'_e/\omega_e$ and in addition $\tau\gamma(\omega'_e/\omega_e, k_\perp) \ll 1$, $\tau\nu_{\text{eff}} \ll 1$. The Landau damping of Langmuir waves in an inhomogeneous plasma proceeds in the following manner (see Fig. 1): the amplitude of the oscillations changes little within time intervals t such that $\tau - t \gg \delta = r^2\omega_e'^2\tau^3$, rapid relaxation of the oscillations takes place, at $|t-\tau| \sim \delta$, and the oscillations can be regarded as fully damped at $t - \tau \gg \delta$. This time dependence is the consequence of the linear increase of $\tilde{k}$ with increasing time. A similar damping pattern is qualitatively observed also for other plasma oscillations that satisfy Eq. (2), if the corresponding decrement depends exponentially on $\tilde{k}$. We present now the complete expression for the Langmuir-oscillation potential:

$$\phi(x, t) = -iR(x, \mathbf{k}_\perp)\,\omega_e(x)(\omega_e'^{\,2} t^2 + k_\perp^2)^{-1/2} \exp(i\Psi), \tag{10}$$

$$i\Psi = -i\omega_e(x)t + \frac{i}{2} r^2\omega_e(t^3\omega_e'^{\,2} + 3k_\perp^2 t) - (\pi/8)^{1/2} \frac{\omega_e}{r|\omega_e'|}$$

$$\times [1 + (k_\perp/t\omega_e')^2]^{1/2} \exp\left[-3 - \frac{1 - 3k_\perp^2 r^2}{2r^2(t^2\omega_e'^{\,2} + k_\perp^2)}\right] \tag{11}$$

where $R(x,\mathbf{k}_\perp)=\int_{-\infty}^{+\infty} \tilde{\rho}(\omega_e(x),\mathbf{k}')e^{ik'_x x}\,dk'_x/k'$. Formula (11) is valid under the condition $t^{-1}\omega_e^{-2} \ll \omega''_e/\omega'^2_e \ll t$. The imaginary terms written out in (11) give the correct value of the oscillation phase if $t \ll \min(r\omega'_e)^{-4/5}\omega_e^{-1/5}$, $\omega_e^{-1}(k_\perp r)^{-1/4}$, whereas the amplitude of the oscillations can be calculated from formula (10) up to time $t \sim \tau$. Figure 2 shows the relaxation in a plasma with a "bell-shaped" density distribution $\omega_e^2=\omega_0^2(1-x^2/d^2)$, under the condition that $k_\perp/\omega'_e \ll t \lesssim \tau$:

$$|\mathbf{E}(x,t)| = \left|\frac{\partial\phi(x,t)}{\partial x}\right| = \omega_0 |R(x)| (1-\xi^2)^{1/2}$$

$$\times \exp\left[-\beta\frac{(1-\xi^2)^{3/2}}{\xi}\exp\left(-3-\frac{1-\xi^2}{2\xi^2\theta^2}\right)\right],$$

where $\xi = |x|/d$, $\theta = v_{Te}t/d$, $\beta = (\pi/8)^{1/2}a\omega_0/v_{Te} \gg 1$. It is seen from the figure that the oscillations have the longest lifetime near the maximum of the concentration (the concentration distribution is shown dashed). We note, however, that if $x \lesssim a/(\omega_0 t)^{1/2}$, then our solution becomes inconvenient, since it is necessary to take higher-order terms into account in the expansion (4).

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[1)]Equation (1) with ω and $\mathbf{k}$ given has, generally speaking, several roots $x_0^{(i)}(\omega,\mathbf{k})$. We assume that all these roots are far enough from one another $|k_x(x_0^{(i)}-x_0^{(j)})| \gg 1\,(i\neq j)$, so that we can confine ourselves to consideration of the field near each of these roots separately.

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