Properties of the endpoint of a multiphonon spectrum

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The properties of the energy spectrum of liquid helium near the sound line $\epsilon = up$ are investigated. At $T = 0$ it is possible to impart to helium energy and momentum only in the region on the (ϵ, p) plane above this line, and as this line is approached the energy should be partitioned among an ever increasing larger number of phonons. The probability of this process, i.e., the imaginary part of the corresponding Green's function, is proportional to $\exp(-5n \ln n)$, where n is the number of the produced phonons.

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It is known that at not too high pressures the energy spectrum of liquid helium has a "decay" character, i.e., at small momenta it takes the form[1)]

$$\epsilon = up + \gamma p^3 \tag{1}$$

with $\gamma > 0$. This leads to the appearance of phonon damping at absolute zero temperature. In its subsequent evolution the spectrum "bends" downward, so that at a certain point $p = p^*$ the spectrum curve crosses the line $\epsilon = up$, whose slope is equal to the speed of sound u as $p \to 0$. As the point p^* is approached, the possibility of decay is only into a larger number of phonons, and at the point $p = p^*$ the damping vanishes.[1] It is of great interest to determine the law governing this vanishing. It is this question which is solved in the present paper.

This problem is a particular case of a more general one. The "sound" line $\epsilon = up$ in the (ϵ, p) plane is the lower limit of the phonon spectrum of helium–at absolute zero temperature it is impossible to impart to the helium energy or momentum in the region below this line by producing any number of phonons. Therefore the dynamic form factor of the liquid, i.e., the probability of inelastic scattering of neutrons with transfer of energy and momentum ϵ and p should therefore vanish on this line. We shall solve precisely this general problem by calculating the imaginary part of the Fourier component of the Green's function $G(X) = -i\langle T\rho(X)\rho(0)\rangle$ made up of the liquid-density operators. It is this imaginary part which is the dynamic form factor.

We calculate the minimum number of phonons that can be produced somewhat above the sound line. It is clear beforehand that the most convenient situation is the one in which the phonon momenta are almost equal in magnitude and direction. Let n be the number of produced phonons and let ω and $\mathbf{k}$ be the energy and momentum of each of them. Then $\epsilon = n\omega, p = nk$ and from the dispersion law (1) we obtain

$$\frac{\epsilon}{n} = \frac{up}{n} + \frac{\gamma p^3}{n^3}$$

so that

$$n = \frac{p^{3/2}\gamma^{1/2}}{(\delta\epsilon)^{1/2}}, \qquad \delta\epsilon = \epsilon - up. \tag{2}$$

The method of solution and the character of the obtained result is easiest to explain using as an example the excitation of an anharmonic oscillator with potential energy

$$u(x) = \frac{m\omega_0^2 x^2}{2} + \beta x^3$$

by an external field of frequency $E \gg \omega_0$. According to Landau, the matrix element of such a process can be calculated quasiclassically, by diverting the integration contour to the complex plane (2). We have for the transition matrix element

$$M \sim \exp\{\int^{\infty} [\sqrt{2m(u-E)} - \sqrt{2mu}]dx. \tag{3}$$

At small β the values significant in the integral are $u \gg E$, so that

$$M \sim \exp\left[-E\int_{x_1}^{x_2} \frac{\sqrt{m}dx}{\sqrt{2u(x)}}\right] \equiv \exp(-E\tau),$$

where

$$\tau = \int_{x_1}^{x_2} \frac{\sqrt{m}\,dx}{\sqrt{2u(x)}}$$

is the imaginary time necessary for the oscillator to reach the point x_2 from the x_1 in the forbidden region. At small β it is possible to calculate τ by using the harmonic approximation and cutting off the integral from below at a value of x_1 such that $u(x_1) \sim E$, and from above at a value x_2 such that $m\omega_0^2 x_2^2 \sim \beta x_2^3$. As a result we have

$$|M|^2 \sim \exp\left(-\frac{E}{\omega_0} \ln \frac{m^3 \omega_0^6}{\beta^2 E}\right).$$

The principal problem of the boundary of the multiphonon spectrum was solved by writing down the Green's function in the form of a functional integral. The liquid Hamiltonian corresponding to the dispersion law (1) is of the form

$$H = \int d^3x \left\{ \frac{\rho_0}{2}(\nabla\phi)^2 + \frac{u^2}{2\rho_0}\rho'^2 + \frac{\gamma u^2}{\rho_0}(\nabla\rho')^2 + \frac{(\nabla\phi)\rho'(\nabla\phi)}{2} + \left(\frac{d}{d\rho_0}\frac{u^2}{\rho_0}\right)\frac{\rho'^3}{6} \right\}, \tag{4}$$

where ρ' and ϕ are respectively the operators of two canonically conjugate quantities—the change of the density of the liquid and the velocity potential. The Green's function of interest to us can be written in the form[3]:

$$G(\epsilon, \mathbf{p}) = -i \frac{\int \{ e^{i(\epsilon t - \mathbf{p}\mathbf{r})} \rho'(t,\mathbf{r})\rho'(0,0)\, e^{iS} \} D\rho' D\phi\, dt\, d^3x}{\int e^{iS} D\rho' D\phi}, \tag{5}$$

where the integration is over the values of ρ' and ϕ at each point of the 4-space $(t,\mathbf{r})$, while S stands for the functional:

$$S = \int \phi \frac{\partial\rho'}{\partial t} d^3x_1\, dt_1 - \int H(\phi, \rho')\, dt_1 . \tag{6}$$

Inasmuch as a large number of phonons take part in the process, it is clear beforehand that a quasiclassical situation is obtained and it is possible to calculate the imaginary part of G by the saddle-point method. The condition for the stationarity of the exponential allows us to replace the functional S in the numerator by its value on the classical trajectories, i.e., by the classical action, and to take it outside the sign of the functional integration. The trajectories on the sections $-\infty < t_1 < 0$ and $t < t_1 < \infty$ should then have the zero total energy and momentum, since ρ' and ϕ should decrease as $t \to \pm\infty$, and its values on the section $0 < t_1 < t$ should be respectively ϵ and $\mathbf{p}$. The condition that ρ' and ϕ be continuous at $t_1=0$ and t can be satisfied then only for trajectories in the classically forbidden region. This means that in the calculation of the action it is necessary to change over to imaginary time, and, just as in the case of an oscillator, the most significant part of the trajectory is the one for which the cubic terms in the Hamiltonian are still small compared with the quadratic terms. This makes it possible to represent the imaginary part of the Green's function in the form

$$\mathrm{Im}\, G \sim \exp(-2\epsilon\tau), \tag{7}$$

where τ is the "imaginary time" necessary for a packet of phonons with frequencies close to ω and with total energy ϵ and momentum $\mathbf{p}$ to acquire a density comparable with the unperturbed density ρ_0. The shape of the packet must be chosen such that the time τ turns out to be minimal. Since the Fourier components of the density increase like $\rho_k \sim e^{uk\tau}$, we have in order of magnitude $\rho(\mathbf{r}) \sim \rho_0 \Delta_{\parallel}^{1/2} \Delta_{\perp} e^{uk\tau}$, where $\Delta_{\parallel}$ and $\Delta_{\perp}$ are respectively the packet widths in k-space in the directions along and across p. It is most convenient to choose $\Delta_{\parallel} \sim (\delta\epsilon)^{1/2}$ and $\Delta_{\perp} \sim \delta\epsilon$. As a result,

$$\tau \approx \frac{5}{2\omega} \ln \left(\frac{\gamma}{\delta\epsilon} \right)^{1/2} \tag{8}$$

and the phonon production probability, i.e., the imaginary part of the Green's function, takes the form

$$\mathrm{Im}\, G \sim \exp(-5n \ln n), \tag{9}$$

where n is the minimum number of the produced phonon and is given by formula (2). Formula (9) is of logarithmic accuracy in the exponent, i.e., the numerical coefficient under the logarithmic sign is intermediate.

[1)]We use a system of units with $h=1$.

[1]L.P. Pitaevskiĭ and I.B. Levinson, Phys. Rev. B **14**, 263 (1976).

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