RESONANT INTERACTION OF WAVE PACKETS IN NONLINEAR MEDIA

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1. The problem of resonant interaction of three wave packets with frequencies ω_1, ω_2, ω_3 and wave vectors $\vec{k}_1$, $\vec{k}_2$, $\vec{k}_3$ is encountered in various physical situations. Two types of interaction are possible in this case — decay and explosive. In decay interaction, the frequencies and wave vectors of the packets satisfy the relations

$$\omega_1 = \omega_2 + \omega_3, \quad \mathbf{k}_1 = \mathbf{k}_2 + \mathbf{k}_3 \tag{1}$$

and the complex envelopes u_1, u_2, and u_3 of the packets in a one-dimensionless nondissipative medium satisfy the system of equations (see, e.g., [1])

$$\begin{aligned} u_{1t} + v_1 u_{1x} &= iq u_2 u_3, \\ u_{2t} + v_2 u_{2x} &= iq u_1 u_3^*, \\ u_{3t} + v_3 u_{3x} &= iq u_1 u_2^*. \end{aligned} \tag{2}$$

"Explosive" interactions of wave packets (see, e.g., [2]) are realized in a medium in which waves with negative energy can exist. The wave vectors and frequencies of the waves satisfy in this case the relations

$$\omega_1 + \omega_2 + \omega_3 = 0, \quad \mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 = 0, \tag{3}$$

while the complex envelopes obey the equations

$$\begin{aligned} u_{1t} + v_1 u_{1x} &= i u_2^* u_3^*, \\ u_{2t} + v_2 u_{2x} &= i u_1^* u_3^*, \\ u_{3t} + v_3 u_{3x} &= i u_1^* u_2^*. \end{aligned} \tag{4}$$

Equations (2) and (4) have been investigated for the case of monochromatic-wave interaction, when $u_i(x, t) = u_i(t)$. A more interesting problem, however, is that of the interaction of wave packets bounded in length, which corresponds directly to the experimental situation in nonlinear optics and plasma physics.

The method of the inverse scattering problem, which has been developed in recent years in connection with problems of wave physics in nonlinear media, makes it po-sible to integrate certain nonlinear partial differential equations without assuming the nonlinearity to be small. This method is applicable in the case when it is possible to construct a pair of nonlinear operatores $\hat{L}$ and $\hat{A}$, such that the investigated equation is identical with the operator relation

$$\frac{\partial \hat{L}}{\partial t} = i[\hat{L}, \hat{A}]. \tag{5}$$

The integration of the initial equation reduces in this case to a solution of the direct and inverse spectral problems for the operator $\hat{L}$. The method of the inverse scattering problem was used for the Korteveg - de Vries equation [3 - 6], for the "parabolic" equation of two-dimensional self-focusing [7, 8], and for the equation of a nonlinear string [9].

The inverse-problem method is applicable also the systems (2) and (4). They are connected with differential matrix operators

$$\hat{L} = ia_i \delta_{ij} \frac{\partial}{\partial x} + K_{ij}(x), \quad \hat{A} = ib_i \delta_{ij} \frac{\partial}{\partial x} + \frac{b_i - b_j}{a_i - a_j} K_{ij}, \tag{6}$$
$$i, j = 1,2,3; \; a_1 > a_2 > a_3, \quad K_{ii} = 0.$$

Substituting $\hat{L}$ and $\hat{A}$ in (5), we obtain

$$(a_i - a_j)\left(\frac{\partial K_{ij}}{\partial t} + v_{ij}\frac{\partial K_{ij}}{\partial x}\right) = i q_0 \sum_k e_{ikj} K_{ik} K_{kj} . \qquad (7)$$

Here $v_{ij} = (a_i b_j - a_j b_i)/(a_i - a_j)$, $q_0 = (a_1 - a_2)v_{12} + (a_2 - a_3)v_{23} + (a_3 - a_1)v_{31}$, e_{ikj} is a fully antisymmetrical tensor of third rank. If we put $K_{ij} = K^*_{ji}$, $K_{12} = (q/q_0)\sqrt{(a_2 - a_3)(a_1 - a_3)}u_2$, $K_{13} = (q/q_0)\sqrt{(a_1 - a_2)(a_2 - a_3)}u_1$, $K_{23} = (q/q_0)\sqrt{(a_1 - a_2)(a_1 - a_3)}u_3$, $v_{13} = v_1$, $v_{23} = v_3$, and $v_{12} = v_1$, then the system (7) goes over into the system (2). An analogous substitution, but with $K_{12} = -K^*_{21}$, $K_{31} = K^*_{13}$, and $K_{23} = -K^*_{32}$ causes the system (7) to go over into the system (4).

2. For monochromatic waves, the system (4) has solutions that become infinite within a short time (e.g., $u_1 = u_2 = u_3 = (1/q)[\exp(i\pi/6)]/(t_0 - t)$ and simultaneously on the entire line $-\infty < x < \infty$. The inverse-problem method makes it possible to obtain a denumerable set of families of exact solutions of the system (4), describing the development of explosive instability in bounded packets. The simplest of these solutions is of the form (a_1, $a_2 > 0$, $a_3 < 0$)

$$K_{12} = -\frac{2\eta}{\sqrt{a_1|a_2|}}\,\frac{e^{\beta_1\eta(x - v_2 t - x_2)}}{\Delta}; \quad K_{13} = -\frac{2\eta}{\sqrt{a_1|a_3|}}\,\frac{e^{\beta_2\eta(x - v_3 t - x_3)}}{\Delta},$$

$$\Delta = 1 - e^{2\beta_1\eta(x - v_2 t - x_2)} - e^{2\beta_2\eta(x - v_3 t - x_3)}; \quad K_{23} = \frac{a_1}{2\eta}K_{12}K_{13}\Delta; \qquad (8)$$

$\eta > 0$; x_2 and x_3 are arbitrary constants, $\beta_1 = \alpha_1^{-1} - \alpha_2^{-1}$, and $\beta_2 = \alpha_1^{-1} - \alpha_3^{-1}$. If $v_2 > v_3$, then the solution (8) becomes infinite at some point at a finite instant of time. The same property is possessed also by other exact solutions (they are analogous, from the formal point of view, to the N-soliton solutions of the Korteveg - de Vries equation [5, 6]). An analysis of these solutions leads to the conclusion that the result of the development of the explosive instability in a medium is the formation of "collapses," which are local singularities of the wave field. If the wave packets are extended (characteristic length $L \gg 1/qu$, where u is the characteristic amplitude), then the number of such collapses is large and they are disposed in disorderly fashion. Under certain special initial conditions, the collapse points can "stick together" and even entire extended collapse segments may be formed.

It is important to note that inclusion of nonlinear frequency shifts does not limit the explosive instability, and merely shifts the collapse points. The amplitude can be limited by nonlinear dissipative effects.

3. In the case of the "decay" system (2), the inverse-problem method makes it possible to solve effectively the problem of wave-packet collision, i.e., to determine the connection between their states that are asymptotic at $t \to \pm\infty$. It turns out here that the physical picture of packet scattering depends significantly on the ratio of their velocities. If the velocity of the wave 1 (which we shall call the pump) is not extremal (this is the situation realized, for example, in the decay of capillary wave on the surface of a liquid into a capillary wave and a gravitational wave), then there is no significant redistribution of energy among the different waves. This effect is particularly strongly pronounced for large and smooth packets ($Lu \gg 1/q$), for which the quantities $I_i = \int_{-\infty}^{\infty}|u_i|^2dx$ are conserved with exponential accuracy. Collision of the packets does lead, however, to a complete change in their shape, and in particular to a broadening of their spectra to a value on the order of qu. The shapes of the produced packets are unstable against small changes in the forms of the colliding waves. It should be noted that the inverse-problem method admits in this case, too, of a set of exact solutions of the system (2). These solutions decay as $t \to \pm\infty$ into non-interacting packets whose integral intensities satisfy the rigorous equality $I_i(-\infty) = I_i(+\infty)$.

On the other hand, if the pump velocity is extremal (as is the case in most experimental situations, particularly in stimulated Mandel'shtam-Brillouin scattering of electromagnetic waves in a plasma), then the picture of the packet scattering is entirely different. If, for example, $v_2 < v_3 < v_1$, then collision of the "large" wave packets 1 and 3 can be accompanied by a complete decay (accurate to exponentially small terms) of the pump, as is indeed the case when the following condition is satisfied:

$$\frac{\max|u_3|^2}{\max|u_1|^2} > \frac{v_1 - v_2}{v_3 - v_2} \,. \tag{9}$$

The produced wave packet 2 has in this case a spectral width $\Delta k \sim q[(v_1 - v_3)/(v_1 - v_2)]^{1/2} \cdot [(\max|u_1|)/(v_3 - v_2)]$. We emphasize that the condition (9), which, however, is valid only for sufficiently smooth packets) does not contain the characteristic dimensions of the packets or the "interaction constant" q. We note also that the energy transfer takes place (in collision of "large" packets) only in the indicated situation, i.e., when there are present in the "input channel" a pump wave and a secondary wave whose velocity is not extremal (wave 3 in our example). In collisions of the "large" packets 1 and 2 or 2 and 3, on the other hand, there is no redistribution of the energy.

We note in conclusion that the system (2), like other systems that are integrable by the inverse-problem method, has an infinite set of integrals of motion, which are simply connected with the scattering matrix of the operator $\hat{L}$ (6).

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INTERFERENCE EFFECTS IN COLLISION OF MAGNESIUM IONS WITH RUBIDIUM AND CESIUM ATOMS

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Experiments on inelastic collisions between rare-earth ions and alkali-metal atoms were performed for the first time in our laboratory. We report here the results on the excitation of certain spectral transitions that occur when magnesium ions collide with rubidium and cesium atoms. The experimental setup and procedure are described in [1].

For each pair of colliding particles, we investigated two inelastic channels:

$$Mg^+ + Rb \begin{cases} \to Mg^+ + Rb^*(5^2P_{3/2}) + 1{,}59\ \text{eV} \\ \to Mg^*(4\,^3S_1) + Rb^+ + 1{,}64\ \text{eV} \end{cases}, \tag{1}$$

$$Mg^+ + Cs \begin{cases} \to Mg^+ + Cs^*(6\,^2P_{3/2}) + 1{,}46\ \text{eV} \\ \to Mg^*(4\,^3S_1) + Cs^+ + 1{,}36\ \text{eV} \end{cases} \tag{2}$$

in the incident-ion energy range from 4 to 1000 eV. Figures 1a and 2a show plots of the effective cross sections for the excitation of the investigated transition against the reciprocal