Properties of superfluid systems with multiple zeros in the spectrum of fermions

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The anomalous behavior of superfluid systems, in which the energy gap vanishes at diabolical points with a multiple topological charge, is analyzed.

The superfluid liquids ^{3}He-A and ^{3}He-A_1 and possibly also certain superconductors[1–3] have zeros in the spectrum of fermion excitations. These zeros lead to singularities in the low-temperature properties of these systems. The most interesting properties, which are similar to the chiral anomaly,[4,5] arise in the fairly common case in which the zero is a diabolical point of the spectrum (a point at which two cones are tangent: a "diabolo"[6]). In other words, the zero is a topologically unremovable point of tangency of two branches of the spectrum,[7–9] in this case a branch of quasiparticles and a branch of quasiholes.[10]

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In certain classes of superfluidity and superconductivity, such as the f state induced by a dipole interaction in ^{3}He-A_1 (Ref. 11) and the $D_6(C_2)$ class of singlet superconductivity, which may occur in UPt_3, the diabolical points are multiple points. As we will see below, the multiplicity of the zero changes the anomalous properties.

In an analysis of a diabolical point it is sufficient to consider simply a two-level Hamiltonian which describes those two branches of the spectrum (a quasiparticle branch and a hole branch) which make contact at the diabolical point (and to ignore other branches). The corresponding Bogolyubov Hamiltonian near a diabolical point $\mathbf{k}_0$ in momentum $(\mathbf{k})$ space thus contains no spin or band indices:

$$H(\mathbf{k}) = \begin{pmatrix} \epsilon(\mathbf{k}) & \Delta(\mathbf{k}) \\ \Delta^*(\mathbf{k}) & -\epsilon(\mathbf{k}) \end{pmatrix} = \vec{\tau}\,\mathbf{m}(\mathbf{k}) . \qquad (1)$$

Here $\epsilon(\mathbf{k})$ is the energy spectrum in the normal state, reckoned from the Fermi surface; $\Delta(\mathbf{k})$ is a momentum-dependent gap; and the components of the vector $\mathbf{m}(\mathbf{k})$ decompose H in two-row Pauli matrices $\vec{\tau}$, so the spectrum of particles and holes is $E_\pm = \pm|\mathbf{m}(\mathbf{k})|$. At the diabolical point itself we have $E_+(\mathbf{k}_0) = E(\mathbf{k}_0) = 0$.

The topological invariant N, which shows the multiplicity of the zero or the multiplicity of the tangency of the two branches, is determined by an integral over a closed surface σ around the diabolical point:

$$N = \frac{1}{8\pi} \int_\sigma dS_i e_{ijk} |\mathbf{m}|^{-3} \left(\mathbf{m} \left[\frac{\partial \mathbf{m}}{\partial k_j}, \frac{\partial \mathbf{m}}{\partial k_k} \right] \right) . \qquad (2)$$

In the simplest realization of diabolical points, with topological charges $\pm N$,

$$\epsilon(\mathbf{k}) = v_F(k - k_F),\ \Delta(\mathbf{k}) = \Delta_0 \left(\frac{k_x + ik_y}{k_F} \right)^N , \qquad (3)$$

the point $\mathbf{k}_{01} = k_F\hat{\mathbf{z}}$ has a charge N, and the point $\mathbf{k}_{02} = -k_F\hat{\mathbf{z}}$ has a charge $-N$. In superfluid ^{3}He, this situation corresponds to Cooper pairing into a state, whose z projection of the orbital angular momentum is $L_z = N$.

The singularities which arise in ^{3}He-A because of diabolical points with $N = 1$ are well known, since in this case Hamiltonian (1) is linear in $\mathbf{k} - \mathbf{k}_0$ near $\mathbf{k}_0 (m_a(\mathbf{k}) = e_a^i (k_i - k_{0i})$ and has a structure which corresponds to the Hamiltonian of a massless chiral electron which is moving in an electromagnetic vector potential[4,10] $\mathbf{A} = \mathbf{k}_0(r,t)$. In the case of superfluid ^{3}He-A_1, quasiparticles with spins parallel to the magnetic field have a diabolical point in the spectrum with[11] $N = 3$, while in the case of the singlet $D_6(C_2)$ superconductivity class we have[2] $N = 2$. It is thus worthwhile to examine the anomalies in the behavior of systems with an arbitrary value of N, in which there is no direct analogy with quantum electrodynamics (QED). In this letter we are presenting the basic results.

1. **Zero-charge effect.**[4] This effect persists at arbitrary N, despite the difference from QED. An expansion of the free energy F in gradients of the position of the diabolical point, $\mathbf{k}_0(\mathbf{r},t)$, or (equivalently) in gradients of the vector $\mathbf{l}(\mathbf{r},t)$, which

specifies the direction of the orbital angular momentum of the Cooper pairs $[\mathbf{k}_0 = \pm k_F \mathbf{l})\mathbf{r},t)]$, has a logarithmic divergence as $T \to 0$:

$$F_3 = K_3 \int d^3 x\, [\mathbf{l}, \mathrm{curl}\, \mathbf{l}]^2 , \qquad K_3 = N \frac{v_F k_F^2}{24\pi^2} \ln \frac{\Delta_0^2}{T^2} \quad . \tag{4}$$

This situation corresponds to QED with N massless charged fermions.

2. **Wess-Zumino action**. For the orbital dynamics of the vector $\mathbf{l}$ (Ref. 12), the Wess-Zumino action, like the Berry phase, can be written in general form in terms of a Green's function $G = (\omega - H)^{-1}$ and an additional coordinate x^5 (Ref. 13, for example):

$$S_{WZ} = \frac{1}{8\pi} \sum_{\mathbf{k}} \int d\omega dt dx^5\, \mathrm{Tr}\,(G \partial_\omega G^{-1} G \partial_t G^{-1} G \partial_5 G^{-1} - t \longleftrightarrow x^5)$$

$$= \frac{1}{4} \sum_{\mathbf{k}} \int dt\, dx^5\, |\mathbf{m}|^{-3} (\mathbf{m}\, [\partial_t \mathbf{m},\ \partial_5 \mathbf{m}\,]\,) \; . \tag{5}$$

This action contains, in addition to the term $(\hbar/2m_3)\rho\dot{\Phi}$ which is natural for superfluidity ($2m_3$ is the mass of the Cooper pair, ρ is the density of the liquid, and Φ is the phase of the condensate), the following contribution from the diabolical points, whose positions $\mathbf{k}_a(\mathbf{r},t,x^5)$ depend on $\mathbf{r},t,x^5$:

$$S_{WZ} = \frac{1}{24\pi^2} \sum_a N_a \int d^3 x dt dx^5\, (\mathbf{k}_a\, [\partial_t \mathbf{k}_a,\ \partial_5 \mathbf{k}_a\,]\,) \; , \tag{6a}$$

where N_a is the topological charge of the a-th point. In the case of two diabolical points of the type in (3), with $\mathbf{k}_a = \pm k_F \mathbf{l}$, the result is the expression found in Ref. 12 for the case $N = 1$, multiplied by N:

$$S_{WZ} = \frac{1}{2} N C_0 \int d^3 x dt dx^5 (\mathbf{l}\, [\partial_t \mathbf{l},\ \partial_5 \mathbf{l}]), \qquad C_0 = \frac{k_F^3}{3\pi^2} \quad . \tag{6b}$$

Because of this additional contribution to the action, the dynamic angular momentum $\mathbf{L}_0$ of the liquid is substantially smaller than its natural value $\hbar N\rho/2m_3$:

$$\mathbf{L}_0 = \frac{1}{2} \hbar N \left(\frac{\rho}{m_3} - C_0 \right) \mathbf{l} . \tag{7}$$

3. **Anomalous current.**[4,5] The anomalous current is also multiplied by N. The same is true of the anomalous current source $\mathbf{I}$, which describes the momentum transfer from the superfluid motion to the normal motion[14,15] and which has a precise analogy with a source of chiral charge in QED[4]:

$$\mathbf{j}_{anom} = -\frac{1}{2} N C_0 \mathbf{l} (\mathbf{l}\, \mathrm{curl}\, \mathbf{l}), \qquad \mathbf{I} = \frac{3}{2} N C_0 \mathbf{l} (\partial_t \mathbf{l} \cdot \mathrm{curl}\, \mathbf{l}). \tag{8}$$

4. **Density of the normal component at T=0**. In the presence of a countercurrent $\mathbf{w} = \mathbf{v}_s - \mathbf{v}_n$, the density of the normal component at $T = 0$ is found from the expres-

sion for the incoherent current of normal excitations. These excitations are described at $T=0$ by a step θ-function distribution: $\mathbf{j}_{\rm inc.}=\Sigma_{\mathbf{k}}\mathbf{k}\theta(\mathbf{kw}-E(\mathbf{k}))$, where the energy E is given by $E^2=\epsilon^2+\Delta_0^2[(\mathbf{k},\mathbf{l})^{2N}/k_F^{2N}]$ according to (3). If $w\ll\Delta_0/k_F$, we have the following expression for the longitudinal density of the normal component, $\rho_{n\|}$:

$$\mathbf{j}_{inc}=\rho_{n\|}\mathbf{l}(\mathbf{lw}),\qquad \rho_{n\|}=N_F k_F^2(k_F(\mathbf{l}\,\mathbf{w})/\Delta_0)^{2/N}\frac{1}{2N}B(1/N,3/2)\ , \tag{9}$$

where N_F is the state density in a normal Fermi liquid.

5. **State density**. It follows from (9) that the state density on the Fermi surface, $N(0)$, is finite in the presence of a countercurrent and is given in order of magnitude by

$$N(0)\sim N_F(k_F(\mathbf{l}\mathbf{w})/\Delta_0)^{2/N}\ . \tag{10}$$

Using (10), we can find $N(0)$ in type-II superconductors in a sufficiently strong magnetic field H, such that the distance between Abrikosov vortices is smaller than the penetration depth. In this case w falls off as $1/r$ outside the core of the vortex, and if the magnetic field is not parallel to the vector **l**, the region between vortices will make the following contribution to the state density:

$$N(0)\sim N_F\frac{H}{H_{c2}}\ln\frac{H}{H_{c2}},\qquad N=1;\qquad N(0)\sim N_F\left(\frac{H}{H_{c2}}\right)^{1/N},\qquad N>1. \tag{11}$$

If, on the other hand, we have $\mathbf{H}\|\mathbf{l}$ and thus $\mathbf{wl}=0$, only the cores of the vortices will contribute to $N(0)$, as in superconductors lacking diabolical points. As a result, $N(0)$ will be a linear function of the magnetic field[16]: $N(0)\sim N_F(H/H_{c2})$. A contribution to the state density linear in H has been observed[17] in UPt_3, but in the case in which **H** was directed along the hexagonal axis, i.e., along a possible vector **l**. Consequently, resolving the question of the existence of and the nature of the zeros will require an experiment with damping of ultrasound in magnetic fields directed obliquely with respect to the axis. The same is true of the specific heat, which would be $\sim T^{1+(2/N)}$ in the absence of a field, while in a field it would depend on both the magnitude and the direction of the field.

6. **State density in the texture of the vector l**. This state density has been found here in the semiclassical approximation, which is the same approach that was taken in Ref. 14 in the case $N=1$, (in the case $N=1$, the result of the semiclassical approximation differs by a factor of 2 from the exact result[15]):

$$N(0)\sim N_F(|[\mathbf{l},\operatorname{curl}\mathbf{l}]|v_F/\Delta_0)^{2/(N+1)} \tag{12}$$

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