

Scattering in two-dimensional model with Lagrangian

$$L = (1/\gamma) \left[(1/2) (\partial_\mu u)^2 + m^2 (\cos u - 1) \right]$$

I. Ya. Aref'eva and V. E. Korepin

Leningrad Division, Mathematics Institute, USSR Academy of Sciences

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It is shown that the fundamental scattering properties of the classical solutions of the equation

$\square u + m^2 \sin u = 0$ are preserved also in the quantum case.

Many solutions of classical nonlinear equations of motion in two-dimensional space-time have been obtained recently.^[1] In particular, solutions were described for a model with an action^[2]

$$A = \frac{1}{\gamma_0} \int d^2x \left[\frac{1}{2} (\partial_\mu u)^2 + m_0^2 (\cos u - 1) \right]. \quad (1)$$

The classical S matrix for this model has the following features: 1) there is no multiple production; 2) the momenta of the final and initial particles are equal; 3) the phase of the n -particle S matrix is equal to the sum of phases of pairwise collisions (factorization).

In this article we show that for a quantum model with action (1) the properties 1 and 2 are preserved. There

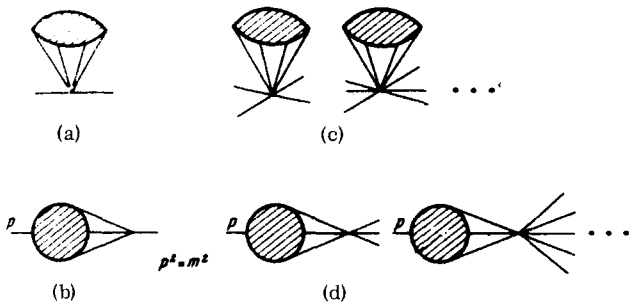


FIG. 1.

are grounds for assuming that property 3 is also preserved. We present here arguments in second-order perturbation theory. We carry out the quantization in standard fashion and calculate the S matrix in accordance with the Feynman rules for the amplitude $F_{n,m}$, where n and m are the numbers of incoming and outgoing particles, respectively. We use perturbation theory in terms of the parameter γ ; the amplitude in order l is designated $F_{n,m}^{(l)}$, and its connected part is designated by $\mathcal{F}_{n,m}^{(l)} \equiv \mathcal{F}_{n,m}^{(l)}$. In the diagrams we draw the amplitude without taking external lines into account. Let us clarify the questions connected with renormalization. The only diverging quantity is $D(0) = \int d^2k (k^2 + m^2)^{-1}$. Mass renormalization corresponds to the diagrams of Figs. 1a and 1b. However, to retain the structure of the interaction (1), we add the counterterms $\Delta L = (\Delta m^2/\gamma)(\cos u - 1)$. It is easy to show that addition of this ΔL corresponds to discarding all the diagrams of the type shown in Fig. 1.

We note that when properties 1 and 2 are satisfied the amplitude \mathcal{F} does not have the usual analytic properties of the scattering amplitudes for realistic models in quantum field theory. Indeed, the properties 1 and 2 are equivalent to the representation $\mathcal{F}_{n,n}$ in the form of a product of $n-2$ delta functions by the reduced amplitude $f_{n,n}$:

$$\mathcal{F}_{n,n} = I f_{n,n}, \quad I = \frac{1}{n!} \sum_{\text{over permut. of } \{a_1, \dots, a_n\}} \prod_{i=1}^n \delta^2(k_{i1} - k_{a_i 1}). \quad (2)$$

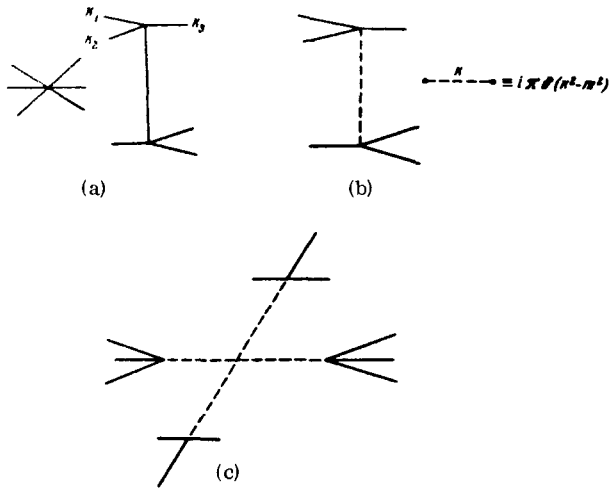


FIG. 2.

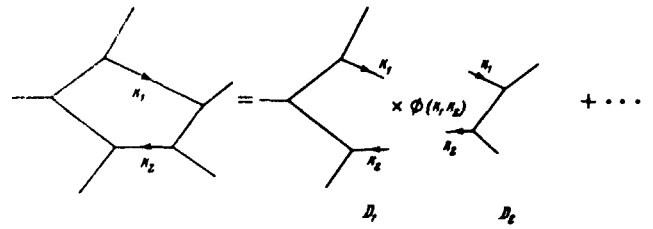


FIG. 3.

We shall call this representation the 1, 2 structure. We assume that the reduced amplitude $f_{n,n}$ has the same analytic properties as $\mathcal{F}_{n,n}$ in the usual models. It is remarkable here that the absence of multiple production does not contradict crossing.

We shall verify the 1, 2 structure by perturbation theory. In the tree approximation, the S matrix is determined by the scattering of the solutions of the classical equations, and therefore the properties 1–3 are preserved. For example, the amplitude $\mathcal{F}_{6,6}^{(-1)}$, calculated from the diagrams of Fig. 2a, vanished in the 2–4 channel, and also in the 3–3 channel, with the exception of the case when the final state coincides with the initial one. In the latter case, in the calculation of the second diagram of Fig. 2a, an uncertainty arises, which we eliminate in the following manner:

$$\frac{1}{m^2 - (k_1 + k_2 - k_3)^2 - i\epsilon} \mathcal{P} \left(\frac{1}{m^2 - (k_1 + k_2 - k_3)^2} \right) + i\pi \delta(m^2 - (k_1 + k_2 - k_3)^2). \quad (3)$$

Equation (3) is valid in the sense of generalized functions, i.e., for the scattering of finite packets. The contribution of the principal values is cancelled by the first diagram of Fig. 2a. The entire amplitude $\mathcal{F}_{3,3}^{(-1)}$ is described by diagram 2b. It is easy to show that any amplitude $\mathcal{F}_{n,n}^{(-1)}$ can be calculated as a sum of trees with quadrupole vertices and dashed in channel lines (Fig. 2b). This leads directly to the 1, 2 structure for $\mathcal{F}_{n,n}^{(-1)}$.

One might assume, however, that $\mathcal{F}_{3,3}$ is an analytic continuation of $\mathcal{F}_{2,4}$ (as is the case in real models), i.e., $\mathcal{F}_{3,3}^{(-1)} = 0$. It can be easily verified, however, that according to perturbation theory the equality $\mathcal{F}_{3,3}^{(-1)} = 0$ contradicts the unitarity condition.

To prove the 1, 2 structure for $\mathcal{F}_{n,n}^{(0)}$, we made use of the fact that in two-dimensional space any single-loop diagram is expressed in terms of elementary functions. We formulate a rule for an arbitrary loop. It is a sum over all possible pairs of internal lines with momenta k_1 and k_2 . We determine k_1 and k_2 from the conditions $k_1^2 = k_2^2 = m^2$. The contribution of the separated pair is equal to two terms, each of which is a product of $\Phi(k_1, k_2)$ by two amplitudes of the tree type, D_1 and D_2 , which are complements of the lines with momenta k_1 and k_2 needed to form the entire loop; $\Phi(k_1, k_2)$ is specified by the expression

$$\text{Re} \left[\frac{1}{8\pi\sqrt{-st}} \ln \frac{\sqrt{-s} - \sqrt{t}}{\sqrt{-s} + \sqrt{t}} \right] + i \frac{\theta(k_{10})\theta(-k_{20}) + \theta(-k_{10})\theta(k_{20})}{4\sqrt{-st}},$$

$$s = (k_1 + k_2)^2, \quad t = (k_1 - k_2)^2.$$

This formula is shown symbolically in Fig. 3. We proceed to verify the 1,2 structure in one loop. It follows from unitarity for $\mathcal{J}_{2n}^{(0)}$ that $\text{Im}\mathcal{J}_{2n}^{(0)}$ has the 1,2 structure. We shall verify that in a certain range of variation of the variables the entire amplitude $\mathcal{J}_{2n}^{(0)}$ has the same structure. We represent each propagator in the right-hand side of the formula in Fig. 3 in the form (3) and break it up into groups, each of which contains terms with identical number of δ functions (from 0 to $n-2$). Each group in some channel has an imaginary part. From unitarity for $\mathcal{J}_{2n}^{(0)}$ it is clear that all the groups, except the one containing $(n-2)$ delta functions, cancel out. Therefore $\mathcal{J}_{2n}^{(0)}$ has the 1,2 structure in some region, and from analyticity considerations it has the same structure also at all values of the parameters.

In l th order of perturbation theory, using unitarity and assuming a 1,2 structure for the lowest orders, we find that $\text{Im}\mathcal{J}_{2n}^{(l)}$ has the same structure. We reconstruct $\text{Re}\mathcal{J}_{2n}^{(l)}$ from the analyticity of the reduced amplitude.

In the absence of multiple production, the unitarity condition for $f_{2,2}$ can be easily solved, and $S_{2,2}$ takes the form

$$\langle k_1 k_2 | S | k_1' k_2' \rangle = i \frac{4\sqrt{st} g(s) + \gamma^2}{4\sqrt{st} g(s) - \gamma^2}, \quad s = (k_1 + k_2)^2, \quad t = 4m^2 - s,$$

where

$$g(s) = A - \sum_n \frac{R_n}{\alpha_n^2 - (s - 2m^2)^2}, \quad R_n > 0, \quad [3].$$

The growing term of 3 is absent, and α_n seem to be located at the thresholds, $\alpha_n = (4n^2 - 2)m^2$. In three

orders of perturbation theory, $g(s)$ is equal to

$$\frac{\gamma}{m^2} - \frac{\gamma^2}{8\pi m^2} - \frac{\gamma^3 m^2}{8s(4m^2 - s)}.$$

As $\gamma \rightarrow 0$, $S_{2,2}(s)$ has two poles in the gap $(0, 4m^2)$, and this can be interpreted as the appearance of a new particle with mass $M^2 = 4m^2 - m^2(\gamma/4)$.

In analogy with the classical case, we assume factorization of the S matrix in terms of the two-particle scattering:

$$S = \exp \left\{ \int \frac{dk_1 dl_1}{k_o l_o} : \rho(k_1) \rho(l_1) : \phi_{2,2}((k+l)^2) \right\}, \quad (4)$$

where

$$\rho(k_1) = a^+(k_1) a(k_1), \quad [a(l_1), a^+(k_1)] = k_o \delta(l_1 - k_1) \text{ and } \phi_{2,2} = \frac{1}{2} \ln S_{2,2}.$$

The S matrix (4) is unitary and has the 1,2 structure. We have verified such a representation in the first two orders of perturbation theory.

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