Above-barrier soliton reflection

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The amplitude of the reflection of a soliton by an antisoliton is calculated in the "sine-Gordon" quantum model. This process is classically forbidden. The potential of soliton interaction in the nonrelativistic limit is calculated.

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The Lagrangian of the model is

$$L = \frac{1}{\gamma} \int dx \left[\frac{1}{2} (\partial_\mu u)^2 - m^2 (1 - \cos u) \right] .$$

The solutions of the classical equation are described in[1,2]. The classical scattering of a soliton (S) by an antisoliton (A) in the c. m. s., which is described by the equation

$$u_{+-}(t, \phi_-) = 4 \operatorname{arc\,tg} \left[\operatorname{cth} \phi_- \frac{\operatorname{sh}(mt \operatorname{sh} \phi_-)}{\operatorname{ch}(mx \operatorname{ch} \phi_-)} \right], \quad p_- = -p_+ = \frac{8m}{\gamma} \operatorname{sh} \phi_- , \tag{1}$$

takes place without reflection and the quasiclassical ($\gamma \to 0$) S-matrix is given by[3,4]

$$< p_+', p_-' | S | p_+, p_- > = \delta(p_+ - p_+')\delta(p_- - p_-')D, \quad |D| = 1,$$

$$D = \exp\left\{ i \frac{8\pi^2}{\gamma} + \frac{8}{\gamma} \int_0^\pi d\theta \ln \frac{\xi e^{-i\theta} + 1}{\xi + e^{-i\theta}} \right\}, \quad \xi = e^{|\phi_- - \phi_+|}, \quad p_\pm = \frac{8m}{\gamma} \operatorname{sh} \phi_\pm . \tag{2}$$

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Here $p_+(p_-)$ is the momentum of $S(A)$. We consider the reflection of S from A in a quantum system in the quasi-classical limit (as $\gamma \to 0$).

The problem of above-barrier reflection in nonrelativistic quantum mechanics was solved in[5,6]. The "imaginary time" method[7,8] turns out to be effective in our problem, and we use also the method of complex trajectories in a functional integral (FI), a method close to that in[9].

The asymptotic form of the propagation function, describing reflection as $t' \to \infty$ and $t \to -\infty$, is given in the p-representation by a regularized FI[3,4] (the region of integration is bounded by constraints and by additional conditions)

$$G(t',t) = \delta(p_+ - p_-')\,\delta(p_- - p_+')N\int_r \exp\{i\int_t^{t'} L dt\}\prod_{x,t} du; \quad u|_t = u_{+-}(t,\phi); \tag{3}$$

$$u|_{t'} = -u_{+-}(t', \phi_-).$$

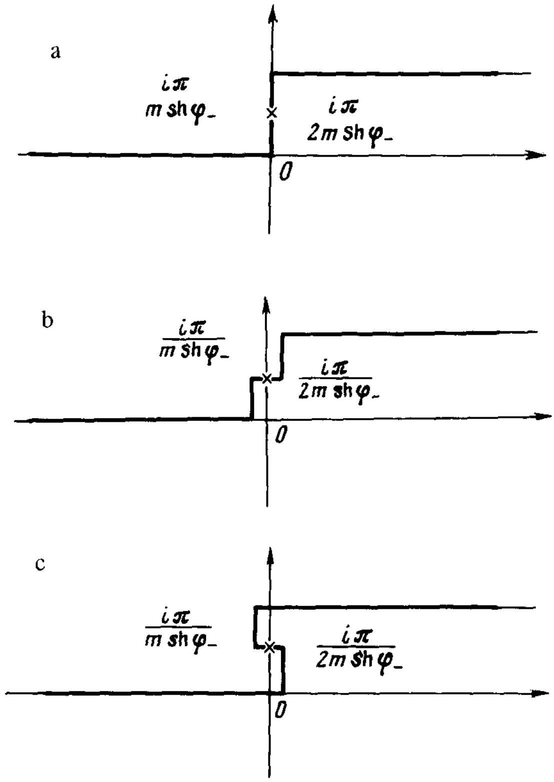
In (3) we have changed over to the c.m.s. The FI in (3) has no stationary-phase point, but if t' is shifted to the complex plane, then such a point does appear. There is a known representation for $G(t',t)$ in the form of a FI with t' and t complex.[10,11] The FI does not depend on the contour that joins t' and t in the complex time plane.[12] We choose t' such that there exists a classical trajectory (CT) corresponding to reflection. It turns out that $\mathrm{Im}\,t' = \pi/m \sinh\phi_-$. We determine the FI (3) along the contour of Fig. a. In this case there exist two CT("b" and "c") that join together the boundary conditions (3). The CT "b" is the limit of values of (1) on the contour of Fig. b, while the CT "c" is the limit of values of (1) on the contour of Fig. c. For both CT, the turning takes place at $t = i\pi/2m \sinh\phi_-$. Calculating the FI (3) by the stationary-phase method, we obtain[9]

$$G(t',t) = -i\exp\{is_b^{cl}\}\det{}_r^{1/2}\frac{\delta^2 s_b^{cl}}{\delta u(t')\delta u(t)} + i\exp\{is_\theta^{cl}\}\det{}_r^{1/2}\frac{\delta^2 s_\theta^{cl}}{\delta u(t')\delta u(t)} \tag{4}$$

$$G(t', t) = \delta(p_+ - p_-')\,\delta(p_- - p_+')\tilde{G}(t', t).$$

Here $s^{cl} = \int_t^{t'} L(u_{+-})\,dt$, while the first (second) term is the contribution of the CT "b" ("c"). In (4), the action is varied with respect to a value of the CT close to "b" or "c" at the instant t' and t. In (4), $-i$ and i are the contributions of the turning points. The explicit form of $\delta_u^{cl}(x,t)$[13] shows that the determinants (4) are equal, and we take them outside the curly brackets and disregard them. The ratio of the contributions of the CT was taken into account by us in the "one-loop" approximation, and the remainder only in the "tree" approximation. We continue $G(t',t)$ in t' back to the real axis. The transition from $G(t',t)$ to the S matrix, just as in quantum mechanics, reduces to replacement of s^{cl} by $s^{cl} - s_+^{cl} - s_-^{cl}$. Here s_+^{cl} and s_-^{cl} are the values of the action calculated on the free S and A, respectively. We ultimately find that the correction to the S matrix (2), corresponding to reflection, is equal to

$$\langle p_+', p_-' | S | p_+, p_- \rangle = -2\delta(p_+ - p_-')\,\delta(p_- - p_+')D\sin\frac{8\pi^2}{\gamma}\exp\left\{-\frac{8\pi}{\gamma}|\phi_- - \phi_+|\right\} \tag{5}$$



At $\gamma = 8\pi/n$ (n is an integer) there is no reflection and decay of the nth bound state of the S and A takes place,[13,3] thus recalling the behavior of a particle in a nonrelativistic potential (NP) $B/\cosh^2(\alpha r)$. The absence of reflection confirms the hypothetical form of the scattering matrix l of S by A at $\gamma = 8\pi/n$[3]:

$$\langle p'_+, p'_- | S | p_+, p_- \rangle = \delta(p_+ - p'_+)\,\delta(p_- - p'_-)\,(-1)^n \prod_{k=1}^{n-1} \frac{\xi e^{-i\theta_k} + 1}{\xi + e^{-i\theta_k}} ,$$

$$\theta_k = \frac{k}{n}\pi .$$

After completing this work, the author has learned of a preprint[14] dealing with the reflection of S and A. In[14], the S and A are replaced by particles that interact with an NP. The quantization reduces to a solution of the Schrödinger equation. The NP was calculated in[14] incorrectly. The reflection coefficient given in[14], which is valid at $\phi_\pm \to 0$, is not obtained, the author's statements notwithstanding, by solving the Schrödinger equation given there.

Let us calculate the NP from the value of the action s_j^{cl} on the CT describing the passage of S and A(1). It obviously gives the correct S matrix (2) as $\phi_+ \to 0$. Let $x_+(x_-)$ and $x'_+(x'_-)$ be the coordinates of $S(A)$ at the instants t and t'. As $t' \to \infty$ and $t \to -\infty$, the form of s_j^{cl} at $\phi_\pm(x_+ - x_-, x'_+ - x_+, x'_- - x_-, t', t) \to 0$ becomes

$$s_f^{cl} - s_+^{cl} - s_-^{cl} = \frac{8\pi^2}{\gamma} + \frac{16}{\gamma} |\phi_- - \phi_+| \ln \frac{|\phi_- - \phi_+|}{2} - \frac{16}{\gamma} |\phi_- - \phi_+| + O\left(\frac{1}{t' - t}\right) + O(|\phi_- - \phi_+|^2 \ln |\phi_- - \phi_+|). \quad (6)$$

The NP made up of the first two terms in (6), which predominate as $\phi_\pm \to 0$, leads to the Schrödinger equation

$$i\frac{\partial\psi}{\partial t} = \left[-\frac{\gamma}{8m}\frac{d^2}{dR^2} - \frac{\gamma}{8m}\frac{d^2}{dr^2} - \frac{2\pi^2 m}{\gamma \operatorname{ch}^2\left(\frac{mr}{2}\right)}\right]\psi, \quad R = x_+ + x_-, \quad r = x_+ - x_-, \quad (7)$$

which describes correctly, as $\gamma \to 0$ and $\phi_\pm \to 0$, the spectrum of the double solitons[3,13] (2) and (5). Vinciarelly[14] calculates the NP by using the dependence of $(p_- - p_+)_{eff}$ on $(x_- - x_+)_{eff}$ along the CT, i.e., actually along $\partial s_f^{cl}/\partial(x_- - x_+)$, in which the principal nonrelativistic term $8\pi^2/\gamma$ is lost, and this indeed leads to an incorrect coefficient in front of $1/\cosh^2(mr/2)$. The CT specify the type of NP only as $r \to \infty$; the function $\exp\{-m|r|\}$ proposed in[15] is indistinguishable from $1/\cosh^2(mr/2)$. We choose (7) because of (5).

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