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Stabilization of Yang–Mills chaos in non-Abelian Born–Infeld theory

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We investigate dynamics of the homogeneous time-dependent $SU(2)$ Yang-Mills fields governed by the non-Abelian Born-Infeld lagrangian which arises in superstring theory as a result of summation of all orders in the string slope parameter α'. It is shown that generically the Born–Infeld dynamics is less chaotic than that in the ordinary Yang–Mills theory, and at high enough field strength the Yang–Mills chaos is stabilized. More generally, a smothering effect of the string non-locality on behavior of classical fields is conjectured.

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The chaotic nature of the classical Yang–Mills (YM) dynamics was recognized more than twenty years ago [1 – 5]. It can be revealed already on simple models of the homogeneous fields depending only on time [6]. Further studies have shown persistence of chaotic features in classical dynamics of spatially varying YM fields leading to a new kind of the "ultraviolet catastrophe" [7].

Here we would like to investigate whether this tendency is preserved on a deeper level of superstring theory, or the latter provides a smothering effect. The possibility to explore this question quantitatively is related to existence of a closed form effective action for the open strings or the stuck of D-branes which accumulates all orders in the string slope parameter α'. Strictly speaking, an exact in α' effective action is known only for the $U(1)$ gauge field, in which case it is the Born–Infeld (BI) action [8, 9]. In the non-Abelian case the exact calculation of the effective action is not possible, but a certain non-Abelian version of the Born–Infeld (NBI) action can still be envisioned in a closed form. This problem was extensively discussed recently [10 – 12], one reasonable proposal (though perhaps still not giving the precise answer in the sense of superstring theory) involves a symmetrized trace [10] construction of the action. Another, simpler form, uses a direct non-Abelian generalization of the $U(1)$ BI action, we will call it the ordinary trace model. Investigation of static solitons in NBI theory [13] has shown qualitative agreement between the results of both the symmetrized trace and the ordinary trace models, so here, for the reason of simplicity, we adopt the latter one:

$$S_{NBI} = \beta^2 \times \int d^4x \left(1 - \sqrt{1 + \frac{1}{2\beta^2} F^a_{\mu\nu} F_a^{\mu\nu} - \frac{1}{16\beta^4} (F^a_{\mu\nu} \tilde{F}_a^{\mu\nu})^2}\right), \tag{1}$$

with β being the critical BI field, in string theory $\beta = 1/2\pi\alpha'$.

The simplest non-Abelian configuration for which the ordinary YM theory predicts chaotic behavior [6] is the following

$$A = u\mathbf{T}_1 dx + v\mathbf{T}_2 dy, \tag{2}$$

where u and v are functions of time only, and $\mathbf{T}_1$,$\mathbf{T}_2$ are the gauge group generators. The corresponding field strength

$$F = \dot{u}\mathbf{T}_1 dt \wedge dx + \dot{v}\mathbf{T}_2 dt \wedge dy + uv\mathbf{T}_3 dx \wedge dy,$$

contains both electric and magnetic components, but these are mutually orthogonal, so the pseudoscalar invariant $\mathrm{tr}\, F\tilde{F}$, generically dominant (1) at high field strength, is equal to zero. The corresponding one-dimensional lagrangian reads

$$L = \beta^2 \left(1 - \sqrt{1 - \beta^{-2} \left(\dot{u}^2 + \dot{v}^2 - v^2 u^2\right)}\right). \tag{3}$$

The equations of motion have the form

$$\ddot{u} = -uv^2 + \frac{2\dot{u}vu(\dot{u}v + \dot{v}u)}{\beta^2 + u^2v^2}, \tag{4}$$

$$\ddot{v} = -vu^2 + \frac{2\dot{v}vu(\dot{u}v + \dot{v}u)}{\beta^2 + u^2v^2}. \tag{5}$$

In the limit $\beta \to \infty$ of the ordinary YM action the second terms in the right hand sides of these equations vanish and one obtains the "hyperbolic billiard" [6, 14, 15], known in various problems of mathematical physics, which is a two-dimensional mechanical system with the potential u^2v^2. This potential has two valleys along the lines $u = 0$ $v = 0$, the particle motion being confined by hyperbolae $uv = \mathrm{const}$. The existence of these valleys is crucial for emergency of chaos.

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The trajectories of the system governed by the BI action depend on the energy integral E and the parameter β. Rescaling of the field variables together with time rescaling maps configurations with different E and β onto each other, so it will be enough to consider dependence of the dynamics only on β assuming for the energy any fixed value, e.g. $E = 1$. One simple (though non-rigorous) method to reveal chaoticity is the analysis of the geodesic deviation equation for a suitably defined pseudoriemannian space whose geodesics coincide with the trajectories of the system [16, 17]. The lagrangian (3) gives rise to the following metric tensor

$$ds^2 = \left(1 + \frac{u^2v^2}{\beta^2}\right) dt^2 - \frac{1}{\beta^2}(du^2 + dv^2), \qquad (6)$$

which is regular for all finite $\beta \neq 0$. Consider the deviation equation for two close geodesics

$$\frac{D^2 n^i}{ds} = R^i{}_{jkl} u^j u^k n^l, \qquad (7)$$

where $D^2 n^i$ is the covariant differential of their transversal separation, $R^i{}_{jkl}$ is the Riemann-Christoffel tensor, and u^j is the three-speed in the space (6). Locally, the geodesic deviation is described by the matrix $R^i{}_{jkl}u^ju^k$ depending on points $u, v, \dot{u}, \dot{v}$ of the phase space. If at least one eigenvalue of this matrix is positive, the geodesics will exponentially diverge with time. Negative eigenvalues correspond to oscillations or to slower divergence, e.g. power-law. In our case one of the eigenvalues of the matrix $R^i{}_{jkl}u^ju^k$ vanishes (due to the static nature of the metric), two others are the roots of a quadratic equation and read

$$\lambda_{1,2} = \frac{1}{2}\left(\mathcal{B} \pm \sqrt{\mathcal{B}^2 + 4\mathcal{C}}\right), \qquad (8)$$

where

$$\mathcal{B} = \frac{2uv\dot{u}\dot{v} - \beta^2(u^2+v^2)}{\beta^2 UW} + \frac{(\dot{v}u + \dot{u}v)^2}{\beta^2 U^2 W}, \qquad (9)$$

$$\mathcal{C} = \frac{v^2u^2(3\beta^2 + u^2v^2)}{\beta^2 U^2 W}, \qquad (10)$$

with W and U being positive functions

$$W = 1 + \frac{u^2v^2 - \dot{u}^2 - \dot{v}^2}{\beta^2}, \qquad U = 1 + \frac{u^2v^2}{\beta^2}.$$

It is easy to see that both non-zero eigenvalues (8) are real, one positive and the other negative. Therefore, for any finite β, there exist locally divergent geodesics, and it can be expected that for any β the motion will remain chaotic. But in the limit $\beta \to 0$ these eigenvalues tend to zero and the analysis becomes inconclusive. Qualitatively, this phenomenon of decrease of the geodesic divergence with decreasing β can be attributed to the lowering of the degree of chaoticity. It is also worth noting that for $\beta \to 0$ the second terms in the equations of motion (4), (5) look like singular friction terms.

Another simple tool in the analysis of chaos is the calculation of the Lyapunov exponents defined as

$$\chi = \lim_{t\to\infty} \frac{1}{t} \log \frac{|\delta x(t)|}{|\delta x(0)|}, \qquad (11)$$

where $\delta x(t)$ is a solution of the linearized perturbation equation along the chosen trajectory. Here x stands for the (four-dimensional) phase space coordinates of the original system and the Cartesian metric norm is chosen. The positive value of χ signals that the close trajectories diverge exponentially with time so the motion is unstable. Stable regular motion corresponds to zero Lyapunov exponents.

The third standard tool is the construction of Poincare sections. Recall that the Poincare section is some hypersurface in the phase space. The phase space for the system (13) is four-dimensional, but the energy conservation restricts the motion to a three-dimensional manifold. Imposing one additional constraint will fix a two-dimensional surface, and we can find a set of points corresponding to intersection of the chosen trajectory with this surface. For the regular motion one typically gets some smooth curves, while in the chaotic case the intersection points fill finite regions on the surface.

Both methods reveal that with decreasing β the degree of chaoticity is diminished, though we were not able to perform calculations for very small β to decide definitely whether the chaos is stabilized or not. But in fact the ansatz (2) is not generic enough from the point of view of the BI dynamics, since the leading at high field strength term $(F\tilde{F})^2$ in (1) is zero for it. So we can hope to draw more definitive conclusions by exploring another ansatz for which the invariant $(F\tilde{F})$ is non-zero. Such an example is provided by a simple axially symmetric configuration of the $SU(2)$ gauge field which is also parameterized by two functions of time u, v [18]:

$$A = \mathbf{T}_1 u dx + \mathbf{T}_2 u dy + \mathbf{T}_3 v dz. \qquad (12)$$

The field strength contains both electric and magnetic components

$$F = \dot{u}(\mathbf{T}_1 dt \wedge dx + \mathbf{T}_2 dt \wedge dy) + \dot{v}\mathbf{T}_3 dt \wedge dz +$$
$$+ u^2 \mathbf{T}_3 dx \wedge dy + uv(\mathbf{T}_1 dy \wedge dz + \mathbf{T}_2 dz \wedge dx),$$

and now the pseudoscalar term $(F\tilde{F})^2$ is non-zero. Substituting this ansatz into the action (1) we obtain the following one-dimensional lagrangian:

$$L_1 = \beta^2(1 - \mathcal{R}),$$

$$\mathcal{R} = \sqrt{1 - \frac{2\dot{u}^2 + \dot{v}^2 - u^2(u^2 + 2v^2)}{\beta^2} - \frac{u^2(2\dot{u}v + \dot{v}u)^2}{\beta^4}}. \tag{13}$$

In the limit of the usual YM theory $\beta \to \infty$ we recover the system which was considered in [18]:

$$L_{YM} = \frac{1}{2}(2\dot{u}^2 + \dot{v}^2 - u^4 - 2u^2v^2). \tag{14}$$

The corresponding potential has valleys along $v = 0$. Like in the case of the hyperbolic billiard with potential v^2u^2, the dynamics governed by the usual quadratic YM lagrangian motion exhibits chaotic character.

However, for the NBI action with finite β the situation is different. One can use again the geodesic deviation method, the corresponding metric being:

$$ds^2 = \left(1 + \frac{u^4 + 2u^2v^2}{\beta^2}\right) dt^2 - \\ - \frac{du^2 + 2dv^2}{\beta^2} - \frac{u^2(2vdu + udv)^2}{\beta^4}. \tag{15}$$

The energy integral is given by the expression

$$E = \frac{\beta^2 + u^4 + 2v^2u^2}{\mathcal{R}} - \beta^2. \tag{16}$$

One can calculate the eigenvalues of the geodesic deviation operator as before. In general one finds that the larger eigenvalue is smaller than one in the case of the previous ansatz (2). Now the regions in the configuration space appear in which both non-zero eigenvalues are negative, this corresponds to locally stable motion. With decreasing β the relative volume of the local stability regions increases, though for every β there still exist the regions of local instability as well. To obtain more definitive conclusions we performed numerical experiments using Lyapunov exponents and Poincare sections. The results look as follows. For large β the dynamics of the system (13) is qualitatively similar to that in the ordinary YM theory (14) and exhibits typical chaotic features. The trajectories enter deep into the valleys along $u = 0$. The Poincare sections consist of the clouds of points filling the finite regions of the plane, while the Lyapunov exponents remain essentially positive. The situation changes drastically with decreasing β. For some value of this parameter, depending on a particular choice of the initial conditions, one observes after a series of bifurcations a clear transition to the regular motion. The points on the Poincare sections $u = 0$

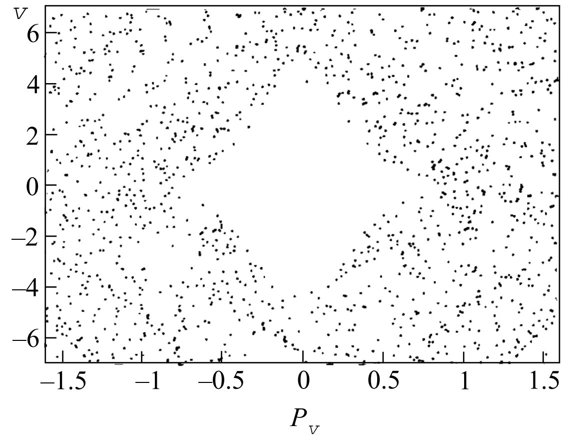

Fig.1. Poincare plane v, P_v of the system (13), where P_v – the canonical momentum conjugate to v, for $\beta = 0.32$

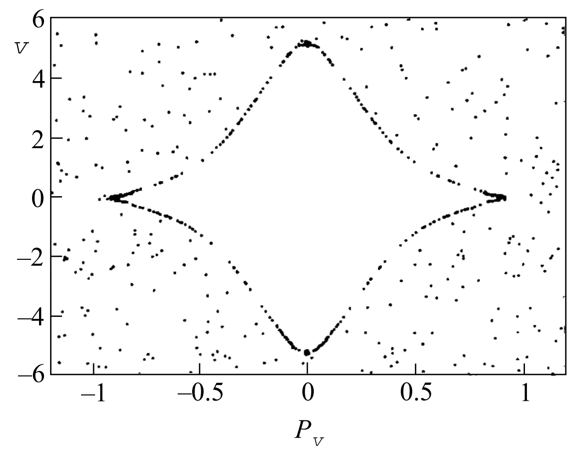

Fig.2. $\beta = 0.317$

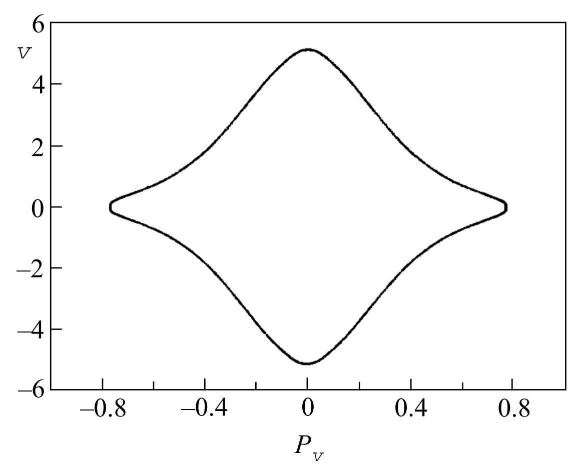

Fig.3. $\beta = 0.31$

line up along the smooth curves (sections of a torus), while the Lyapunov exponent goes to zero. The motion becomes quasiperiodic. A typical feature of such regular motion is that the trajectory no more enters the potential valleys. Some Poincare sections illustrating this chaos-order transition are presented on Figs.1–3. The Lyapunov exponents corresponding to the first and the last picture of this series are shown on Figs.4 and 5 respectively.

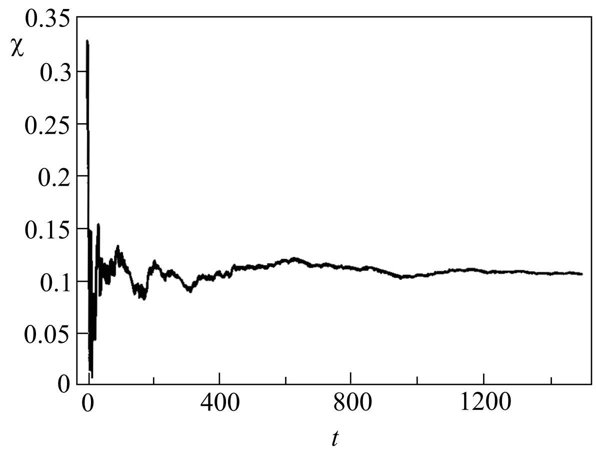

Fig.4. Lyapunov's exponent corresponding to Fig.1

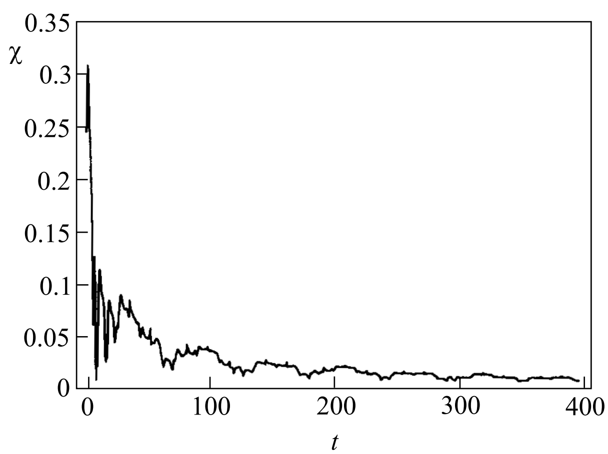

Fig.5. Lyapunov's exponent corresponding to Fig.3

Comparing with our first system (2) we can guess that the substantial role in the chaos-order transition in the NBI theory is played by the pseudoscalar term in the action (1). Since generically this term does not vanish, it is tempting to say that the transition observed is a generic phenomenon in the non-Abelian BI theory. The effect is essentially non-perturbative in α', and we conjecture that it reflects the typical smothering effect of the string non-locality on the usual stiff field-theoretical behavior. A related question is whether the cosmological singularity, which was recently shown to be chaotic at the supergravity level of string models [19], will be also regularized when higher α' corrections are included. Unfortunately there is no closed form effective action for the closed strings analogous to the BI action for the open strings. But combining gravity in the lowest order in α' with an exact in α' matter action we can probe the nature of the cosmological singularity too. This will be reported in a separate publication.

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1. G. Z. Baseian, S. G. Matinian, and G. K. Savvidy, Pisma Zh. Eksp. Teor. Fiz. **29**, 641 (1979).
2. B. V. Chirikov and D. L. Shepelyansky, JETP Lett. **34**, 163 (1981).
3. B. V. Chirikov and D. L. Shepelyansky, Yad. Phys. **36**, 1563 (1982).
4. A. A. Birò, S. G. Matinyan, and B. Müller, *Chaos and Gauge theory*, World Scientific, Singapore, 1994.
5. Sergei G. Matinian, gr-qc/0010054.
6. S. G. Matinian, G. K. Savvidy, and N. G. Ter-Arutunian Savvidy, JETP **53**, 421 (1981).
7. H. B. Nielsen, H. H. Rugh, and S. E. Rugh, chao-dyn/9605013.
8. E. S. Fradkin and A. A. Tseytlin, Phys. Lett. **B163**, 123 (1985).
9. R. G. Leigh, Mod. Phys. Lett. **A4**, 2767 (1989).
10. A. A. Tseytlin, Nucl. Phys. **B501**, 41 (1997), hep-th/9701125.
11. A. A. Tseytlin, hep-th/9908105.
12. S. Gonorazky, F. A. Schaposnik, and G. A. Silva, Phys. Lett. **B449**, 187 (1999), hep-th/9812094.
13. V. V. Dyadichev and D. V. Gal'tsov, Nucl. Phys. **B590**, 504 (2000), hep-th/0006242.
14. G. M. Asatrian and G. K. Savvidy, Phys. Lett. **A99**, 290 (1983).
15. P. Dahlqvist and G. Russberg, Phys. Rev. Lett. **65**, 2837 (1990).
16. M. Szydlowski, Phys. Lett. **A176**, 22 (1993).
17. Y. Sota, S. Suzuki, and K. Maeda, Class. Quantum Grav. **13**, 1241 (1996), gr-qc/9505036.
18. B. K. Darian and H. P. Kunzle, Class. Quant. Grav. **13**, 2651 (1996), gr-qc/9608024.
19. T. Damour and M. Henneaux, Phys. Rev. Lett. **85**, 920 (2000), hep-th/0003139.