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Materializing Superghosts

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We construct the off-shell Batalin-Vilkovysky (BV) realization of $\mathcal{N}=1$, $d=10$ Super-Yang-Milk (SYM) with 7 auxillary fields. This becomes possible due to materialized ghost phenomenon. Namely, supersymmetry ghosts are coordinates on a manifold B of 10-dimensional spinors with pure spinors cut out. Auxillary fields are sections of a bundle over B, and supersymmetry transformations are nonlinear in ghosts. By integrating out axillary fields we obtain on-shell supersymmetric BV action with terms quadratic in antifields. Exactly this on-shell BV action was obtained in our previous paper after integration out of auxiliary fields in the framework of Pure Spinor Superfield Formalism.

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1. What is materialization? Recent investigations of what we should call by symmetry of the system reveal that all empirical notions may be expressed in terms of the very powerful Batalin-Vilkovysky (BV) formalism. Moreover, we believe that this formalism should be considered as a guiding line in extension of the very notion of what we call symmetric system. In our previous paper [1] we explained how BV treats what one may call on-shell symmetry (with control of terms proportional to the equations of motions) on the example of supersymmetry. In the present paper we will explain the new phenomena – the materialization of the superghosts – again in BV formalism. In one sentence it means replacing

$$\delta\phi^i \to \epsilon^a v_a^i(\phi) \qquad (1)$$

by

$$\delta\phi^i \to v^i(\epsilon,\phi), \qquad (2)$$

where v is a homogeneous function of degree 1 in ϵ, not necessarily linear. If we consider ϵ as a parameter (as in the standard "Lie group" way of treating supersymmetry) non-linear expressions in ϵ have no sense. In BV philosophy ϵ should be treated as a ghost rather than as infinitesimal parameter. This difference is essential – nonlinear functions of degree 1 are different from linear ones only if ϵ is *even*, it means that the ghost point of view turns out to be interesting for *supersymmetry*. Perhaps, that is why nonlinear effects what we are discussing here were missed before.

In which sense the transformations (2) close? Traditionally, people would advocate

$$\left\{v^i(\epsilon,\phi)\frac{\partial}{\partial\phi^i}, v^i(\epsilon',\phi)\frac{\partial}{\partial\phi^i}\right\} = (\epsilon\gamma^\mu\epsilon')P_\mu. \qquad (3)$$

Here P_μ is the generator of translations, and γ_μ are standard γ-matrices. Here we stress that the proper expression is a little bit different

$$\left\{v^i(\epsilon,\phi)\frac{\partial}{\partial\phi^i}, v^i(\epsilon,\phi)\frac{\partial}{\partial\phi^i}\right\} = (\epsilon\gamma^\mu\epsilon)P_\mu \qquad (4)$$

(there is no ϵ' in (4)). Expression (4) is the **essence** of ghost materialization approach.

Note, that (3) and (4) are equivalent for linear transformations while they are not equivalent for nonlinear ones. That is why the systems of equations coming from (3) and (4) are different. Authors of [2] find 9-dimensional space of supersymmetries, following the standard approach (3). While following the materialized ghost approach it is possible to reconstruct the full 16-dimensional space of solutions.

Nonlinear equations arising from (4) mean quite an interesting thing – the ghost ϵ turns out to be a point on a manifold of nontrivial topology rather than a linear coordinate on the algebra (with reversed parity).

That is why ghosts enter the game on the equal footing with the matter fields, they are bosonic and span the manifold of possibly nontrivial topology (i.e. they are getting shape). That is why we are calling this phenomenon *materialization*.

1.1. *BV approach to the notion of symmetry.* According to BV approach the symmetric system means that the BV action $S(\Phi,\Phi^*,c,c^*)$ of the special form solves the master equation

$$\frac{\delta S}{\delta c}\frac{\delta S}{\delta c^*}+\frac{\delta S}{\delta \Phi}\frac{\delta S}{\delta \Phi^*}=0.$$

Note, that before indication of the exact form of S the very distinction of variables on ghosts and matter fields is meaningless.

The standard *off-shell symmetric system* means just that

$$S^{\mathrm{off}}(\Phi,\Phi^*,c,c^*) \ = \ S^m(\Phi)+c^a v_a^i(\Phi)\Phi_i^*+f_{ab}^c c^a c^b c_c^*.$$

Here S^m is the invariant action, f_{ab}^c are structure constants of the algebra and v – vector fields, representing the Lie (super)algebra.

The *on-shell symmetric system* is just

$$S^{\mathrm{on}}(\phi,\phi^*,c,c^*)=S^m(\phi)+c^a v_a^i(\phi)\phi_i^*+ \\ +f_{ab}^c c^a c^b c_c^*+\pi_{ab}^{ij}c^a c^b \phi_i^*\phi_j^*,$$

i.e. consists of terms, quadratic both in ghosts and in antifields (the fields Φ and ϕ are different).

The on-shell symmetry corresponds to the closeness of symmetry algebra up to terms proportional to the equations of motion. The common way to get on-shell symmetry is to start with the off-shell symmetrical systems (with fields Φ) and integrate out auxiliary fields (ϕ_{aux}), so that we will stay with the on-shell fields ϕ. However, it may happen that either the off-shell formulation is difficult to find or it involves too many (infinitely many) auxiliary fields.

We are trying to find the set of auxiliary fields and supersymmetry transformations on them, such that there will be no terms quadratic in antifields in the BV action. It happens that this can be done adding 7 auxiliary scalar fields to the classical part of BV action [2]. After that the action becomes linear in antifields, however the supersymmetry part is non-linear in superghosts.

$$S^{\mathrm{GM}}(\Phi,\Phi^*,c,c^*)=S^m(\Phi)+v(c,\Phi)^i\Phi_i^*+f_{ab}^c c^a c^b c_c^*. \tag{5}$$

We call this action the theory with **ghost materialization**. Surely, integrating out these auxiliary fields we get back the on-shell action.

Below we will demonstrate all this phenomena in the case of celebrated $\mathcal{N}=1$, $d=10$ SYM theory.

1.2 *Peculiarities of ghost materialization in $\mathcal{N}=1$, $d=10$ SYM: vector bundle of auxiliary fields.*

It turns out that for $\mathcal{N}=1$, $d=10$ SYM the ghost materialization works not for all but for almost all ghosts that are even 16-dimensional left spinors of $SO(10)$. Namely, the domain B where is works is obtained from $\mathbb{C}^{16}$ by excluding pure spinors, i.e. those that satisfy

$$B=\mathbb{C}^{16}-P,\quad P=\{\varepsilon;(\varepsilon\gamma^\mu\varepsilon)=0\} \tag{6}$$

This is a first manifestation of materialization phenomenon – ghosts form a nontrivial manifold B rather than the linear space.

Moreover, auxiliary fields are sections of the 7-dimensional vector bundle $A\to B$, taking values in the 10-dimensional fields in adjoint representation of the gauge group. We will denote them as $G_i(x)$, $i=1,\dots,7$. The bundle A is equipped with the scalar product $(\cdot,\cdot)_A$ which will be described in the next section. Here we will just show the final result - the BV action for $\mathcal{N}=1$, $d=10$ SYM coupled to auxiliary fields and materialized ghosts[1]:

$$\begin{aligned}S^{MSG}=&\int d^{10}x\,\mathrm{Tr}\Big(-\frac{1}{2}F_{\mu\nu}^2+i\psi\gamma^\mu D_\mu\psi+(G,G)_A-\\&+(D_\mu c)A_\mu^*+g\{\psi,c\}\psi^*-g([G,c],G^*)_A+gccc^*+\\&+i(\varepsilon\gamma^\mu\psi)A_\mu^*\ -\frac{1}{2}(\varepsilon\gamma^{\mu\nu}\psi^*)F_{\mu\nu}+(G,\ \psi^*)_{10}+\\&+i(G^*,\ \gamma^\mu D_\mu\psi)_{10}+\eta^\mu[(\psi^*\partial_\mu\psi)+A_\nu^*\partial_\mu A^\nu+\\&+c^*\partial_\mu c+(G^*,\partial_\mu G)_A]+i\eta_\mu^*(\varepsilon\gamma^\mu\varepsilon)+ic^*A_\mu(\varepsilon\gamma^\mu\varepsilon)\Big).\end{aligned} \tag{7}$$

Integrating out auxiliary fields we get the on-shell action [1] with the terms quadratic in antifields

$$-\frac{1}{8}(\varepsilon\gamma^\mu\varepsilon)(\psi^*\gamma_\mu\psi^*)+\frac{1}{4}(\varepsilon\psi^*)^2,$$

that was obtained in the approach of [1] using the action

$$\begin{aligned}S^{SUSY}=&\int \mathrm{Tr}\Big(\langle\mathcal{P},(Q+\Phi)\mathcal{A}\rangle+g\langle\mathcal{P},\mathcal{A}^2\rangle+\\&+\sqrt{-i}\langle\mathcal{P},\varepsilon^\alpha Q_\alpha^s\mathcal{A}\rangle+\langle\mathcal{P},\eta^\mu P_\mu^s\mathcal{A}\rangle+i\eta_\mu^*(\varepsilon\gamma^\mu\varepsilon)\Big),\end{aligned} \tag{8}$$

with subsequent Z_2 projection (for details see [1]). This action is of the standard superfield type, however, it contains infinitely many auxiliary fields and Z_2 symmetry

[1] We would like to emphasize that the BV form on the fields G and G^* is not unity. The contribution into BV equation is given by

$$\frac{\delta_L S}{\delta G^\alpha}\Big(\frac{1}{2}(\varepsilon\gamma^\mu\varepsilon)\gamma_\mu^{\alpha\beta}\ -\ \varepsilon^\alpha\varepsilon^\beta\Big)\frac{\delta_R S}{\delta G^\beta}.$$

The definitions of the scalar products $(\cdot,\cdot)_{10}$ and $(\cdot,\cdot)_A$ can be found in the next section (see also appendix).

of the effective action is accidental for the present understanding. That is why we use the materialized ghost approach in this paper. We do believe that the geometry of the materialized ghost approach would show up in the study of supergravities, where ghosts are promoted to fields.

2. Geometry of the A-bundle. Let us start with the trivial vector bundle $\mathbb{C}^{16} \to B$ and define its subbundle A as the space of solutions to equations:

$$A = \{a \in \mathbb{C}^{16}, \ \ (a\gamma^{\mu}\varepsilon) = 0\}. \tag{9}$$

Here we have 10 equations on 16 variables, so naively, we may expect 6-dimensional space of solutions. However, one may show that for $\varepsilon \in B$ the space of solutions is really 7 dimensional, and this dimension jumps for pure spinors. Therefore, pure spinors are excluded from the base.

For spinors from the base we will consider the 10-dimensional vector $V(\varepsilon)$:

$$V^{\mu} \ = \ (\varepsilon\gamma^{\mu}\varepsilon).$$

From the Fiertz identity on 10-dimensional gamma-matrixes

$$(\gamma^{\mu})_{\alpha\beta}(\gamma_{\mu})_{\delta\sigma} = -\frac{1}{2}(\gamma^{a})_{\alpha\delta}(\gamma_{a})_{\beta\sigma} - \frac{1}{24}(\gamma^{abc})_{\alpha\delta}(\gamma_{abc})_{\beta\sigma}. \tag{10}$$

we obtain that V is lightlike:

$$V^{\mu}V^{\mu} = 0,$$

also

$$V^{\mu}\gamma^{\mu}\varepsilon = 0 \tag{11}$$

and for any element a of the bundle A

$$V^{\mu}\gamma^{\mu}a = 0. \tag{12}$$

To see this one should contract the r.h.s. of (10) with $\varepsilon^{\alpha}\varepsilon^{\delta}a^{\beta}$. Define a trivial bundle E as

$$E \ = \{\varepsilon \in \mathbb{C}^{16}\}.$$

Therefore, both the bundle A and line bundle E are subbundles of the 8-dimensional bundle C given by:

$$C = \{s \in \mathbb{C}^{16}, \ \ (\varepsilon\gamma^{\mu}\varepsilon)\gamma^{\mu}s = 0\}. \tag{13}$$

For the proof that the bundle C is 8-dimensional see appendix. Moreover, the bundle C comes equipped with the non-degenerate scalar product

$$(s_1, s_2)_C = (s_1, \gamma^{\mu}U^{\mu}s_2)_{10},$$

where for any non-pure spinor ε we define vector U^{μ}, such that

$$U^{\mu}V^{\mu} = 1 \tag{14}$$

and by $(\cdot,\cdot)_{10}$ we denote the standard bilinear paring on 16-component $SO(10)$ spinors (for the definition in terms of anticommuting variables see appendix)

$$(\cdot,\cdot)_{10}: \ \ S^R \otimes S^L \to \mathbb{C}. \tag{15}$$

From the construction it is clear that A is just the orthogonal complement to E in C:

$$C = E \oplus A,$$

$$a \in A, \varepsilon \in E : (a,\varepsilon)_C = 0.$$

Therefore, the pairing $(\cdot,\cdot)_C$ induces on A the pairing $(\cdot,\cdot)_A$.

3. Relation to Berkovits proposal. In order to make contact with Berkovits proposal started in [2] we will consider a patch in the base B where the bundle A may be trivialized. We will look for the orthogonal basis in A that we will call $v_i(\varepsilon)$ considered as elements of $\mathbb{C}^{16}$, and write

$$G = G_i(x)v_i(\varepsilon), \quad G^* = G_i^*(x)v_i(\varepsilon)$$

therefore, the action will take the form

$$\begin{aligned} &S^{MSG} = \\ &= \int d^{10}x \mathrm{Tr}\Big(-\frac{1}{2}F_{\mu\nu}^2 + i\psi\gamma^{\mu}D_{\mu}\psi + G_i^2 - D^{\mu}cA_{\mu}^* + \\ &+ g\{\psi,c\}\psi^* \ - g[G^i,c]G_i^* + gccc^* + i(\varepsilon\gamma^{\mu}\psi)A_{\mu}^* - \\ &- \frac{1}{2}(\varepsilon\gamma^{\mu\nu}\psi^*)F_{\mu\nu} + G_i v_i\psi^* - iv_i\gamma^{\mu}D_{\mu}\psi G_i^* + \\ &+ \eta^{\mu}[(\psi^*\partial_{\mu}\psi) + A_{\nu}^*\partial_{\mu}A^{\nu} + c^*\partial_{\mu}c + G_i^*\partial_{\mu}G_i] + \\ &+ i\eta_{\mu}^*(\varepsilon\gamma^{\mu}\varepsilon) \ + \ ic^*A_{\mu}(\varepsilon\gamma^{\mu}\varepsilon)\Big). \end{aligned} \tag{16}$$

The classical invariant action is given by the first three terms in the first line. The supersymmetry transformations can be extracted from the first fourth terms of the second line. They are exactly those suggested in [2]. The last term in the action reflects [1] the fact that we a working in a certain gauge (analog of Wess-Zumino gauge), thus the commutator of two supersymmetry transformations is closed only up to a gauge transformation with parameter $(\varepsilon\gamma^{\mu}\varepsilon)A_{\mu}$. This action (16) can be considered as an off-shell BV formulation of $\mathcal{N}=1$, $d=10$ SYM.

From the definition of v it follows that

$$\begin{aligned} &v^i\gamma^{\mu}\varepsilon = 0, \\ &v^i\gamma^{\mu}v^k - \delta^{ik}(\varepsilon\gamma^{\mu}\varepsilon) = 0. \end{aligned} \tag{17}$$

Note, that from completeness of the basis it follows that

$$\sum_i v_i^\alpha v_i^\beta = \frac{1}{2}(\varepsilon\gamma^\mu\varepsilon)\gamma_\mu^{\alpha\beta} - \varepsilon^\alpha\varepsilon^\beta. \qquad (18)$$

It should be mentioned that the natural group of symmetry of the system (17) and completeness identity (18) is not $SO(7)$ but $SO(8)$. Namely it is natural to unite ghosts ε and v^i into a single $SO(8)$ multiplet $u^A = (\varepsilon, v^i)$. The system (17) can be written then as

$$u^A\gamma^\mu u^B = \frac{1}{8}\delta^{AB}(u^C\gamma^\mu u_C) \qquad (19)$$

In supergravity ε becomes a field. Thus, probably there will be a local symmetry mixing the ghosts ε and v^i.

Integrating out auxiliary fields G_i from the action (16) on the lagrangian submanifold $G_i^* = 0$, one can obtain the on-shell BV action

$$S^{\text{on-shell}} = \\ = \int d^{10}x \,\mathrm{Tr}\Big(-\frac{1}{2}F_{\mu\nu}^2 + i\psi\gamma^\mu D_\mu\psi - D^\mu c A_\mu^* + \\ +g\{\psi,c\}\psi^* + gcc c^* + i(\varepsilon\gamma^\mu\psi)A_\mu^* - \frac{1}{2}(\varepsilon\gamma^{\mu\nu}\psi^*)F_{\mu\nu} + \\ +\eta^\mu[(\psi^*\partial_\mu\psi) + A_\nu^*\partial_\mu A^\nu + c^*\partial_\mu c] + i\eta_\mu^*(\varepsilon\gamma^\mu\varepsilon) + \\ + ic^*A_\mu(\varepsilon\gamma_\mu\varepsilon) - \frac{1}{8}(\varepsilon\gamma^\mu\varepsilon)(\psi^*\gamma_\mu\psi^*) + \frac{1}{4}(\varepsilon\psi^*)^2\Big) \qquad (20)$$

found in [1] integrating out auxiliary fields from the superfield-like action (8).

Now we are coming to the main point – what is the difference between our approach and approach of [2]? The difference is in the setup. In [2] it was found linear solution to the system (17). Such solution can not be found for the full 16-dimensional space. This is clear from the structure of relations on pure spinor constraints. Namely, in the paper [3] it was shown that the Q-cohomologies can be calculated using the tower of fundamental relations. In case of 10-dimensional quadrics the unique system of these relations is given by

$$\begin{aligned} f^\mu &= \varepsilon\gamma^\mu\varepsilon, \\ G_\alpha^\mu &= (\varepsilon\gamma^\mu)_\alpha, \\ G^{\alpha\beta} &= \varepsilon^\alpha\varepsilon^\beta - \tfrac{1}{2}(\varepsilon\gamma^\mu\varepsilon)(\gamma^\mu)^{\alpha\beta}, \\ G_\alpha^\mu &= (\varepsilon\gamma^\mu)_\alpha, \\ f^\mu &= \varepsilon\gamma^\mu\varepsilon. \end{aligned}$$

This means that the following relations take place: $G_\alpha^\mu f_\mu = 0$, $G^{\alpha\beta}\, G_\beta^\mu = 0$, etc. These relations are valid without imposing pure spinor constraints $f^\mu = 0$. Appearance of 5 relations leads to 6 well known representatives of cohomologies.

Be there a linear solution to the system (17), the first equation

$$v^i(\varepsilon)\gamma^\mu\varepsilon = 0$$

states that there is another relation on G_α^μ, which is linear in ε^α, hence does not coincide with $G^{\alpha\beta}$. If that is true, the cohomologies of SYM would be different. Thus, there is no such linear solution.

The situation is different in $d = 4, 6$, where we do have a linear dependence $v^i(\varepsilon)$ but the structure of cohomologies for SYM is also different [4].

Appendix

The algebra of γ-matrices

$$\{\gamma^\mu,\, \gamma^\nu\} = 2g^{\mu\nu}$$

can be represented by differential operators

$$\gamma^\mu = (\xi^\mu + \frac{\partial}{\partial\xi^\mu}), \qquad \mu = 1\ldots 5, \\ \gamma^{\mu+5} = (-i)(\xi^\mu - \frac{\partial}{\partial\xi^\mu}), \qquad \{\xi^\mu, \xi^\nu\} = 0$$

of odd variables ξ^μ. These operators act between the spaces of left and right spinors which are odd and even functions of ξ^μ respectively. This space is equipped with the Lorentz invariant scalar product

$$(\psi,\, \chi)_{10} = \int d\xi^1 \ldots d\xi^5 i^{P(\chi)} \;\psi\chi,$$

were $P(\chi)$ is the number of ξ^μ variables in the χ.

To solve the system of equations

$$u^A\gamma^\mu u^B = \delta^{AB}(\varepsilon\gamma^\mu\varepsilon) \qquad (21)$$

which is equivalent to (17) and (19) one can use the fact that the vector $(\varepsilon\gamma^\mu\varepsilon)$ is lightlike and choose the frame were

$$(\varepsilon\gamma^1\varepsilon) = 1, \quad (\varepsilon\gamma^6\varepsilon) = i, \quad (\varepsilon\gamma^I\varepsilon) = 0, \quad I = 2...5, 7...10. \qquad (22)$$

Using the definitions

$$\gamma^+ = \frac{1}{2}(\gamma^1 + i\gamma^6), \quad \gamma^- = \frac{1}{2}(\gamma^1 - i\gamma^6)$$

one can write the system (21) as

$$u^A\gamma_- u^B = \delta^{AB}, \quad u^A\gamma_+ u^B = 0, \quad u^A\gamma_I u^B = 0.$$

Since $\gamma^- = \partial/\partial\xi^1$, one can write explicit solution of this system as

$$u_A = \xi^1 P_A(\hat\xi), \quad \hat\xi = \xi^2 \ldots \xi^5. \qquad (23)$$

The space of P_A is 8 dimensional. The corresponding orthonormal basis elements are given by

$$P^+ = \frac{1}{\sqrt{2}}(1 + \xi^2\xi^3\xi^4\xi^5), \quad P^- = \frac{i}{\sqrt{2}}(1 - \xi^2\xi^3\xi^4\xi^5)$$

$$P_K^+ = \frac{i}{\sqrt{2}}(\xi^2\xi^K + \partial_K(\xi^3\xi^4\xi^5)),$$

$$P_K^- = \frac{1}{\sqrt{2}}(\xi^2\xi^K - \partial_K(\xi^3\xi^4\xi^5)), \quad K = 3, 4, 5$$

The last thing to mention is that the C-bundle (13) is 8 dimensional. This is clear from its definition in the light-cone frame (22) $\gamma^+ s = 0$ The general solution of this equation is given by (23), hence is 8-dimensional.

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