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On an explicit construction of Parisi landscapes in finite dimensional Euclidean spaces

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We construct a N-dimensional Gaussian landscape with multiscale, translation invariant, logarithmic correlations and investigate the statistical mechanics of a single particle in this environment. In the limit of high dimension $N \to \infty$ the free energy of the system in the thermodynamic limit coincides with the most general version of Derrida's Generalized Random Energy Model. The low-temperature behaviour depends essentially on the spectrum of length scales involved in the construction of the landscape. We argue that our construction is in fact valid in any finite spatial dimensions $N \geq 1$.

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The idea of energy landscapes pervades the theoretical description of glasses, disordered systems, proteins, etc. [1]. The general goal is to classify typical random potentials and establish their universal properties, not unlike the Random Matrix Theory paradigm. This knowledge can then hopefully be used to describe generic static and dynamic properties of complex systems, by addressing single point particle behavior in such potentials. In this respect, the Parisi solution for spin-glasses is fascinating: it reveals landscapes with a surprisingly complex, hierarchical structure of valleys within valleys within valleys, etc. [2]. It is often however argued that the ultrametric properties of Parisi landscapes are hardly compatible with a finite dimensional, translation invariant space.

In this paper we provide an explicit construction of a Gaussian random potential in Euclidean, N-dimensional spaces, with a specific form of long-ranged correlations which reproduces all the features of Parisi landscapes. More precisely, we show that the thermodynamics of a single particle in a multiscale, logarithmically correlated potential is exactly described by Derrida's Generalized Random Energy Model (GREM, [3]), with an arbitrary (possibly infinite) number of levels of hierarchy. Although our proof concerns, strictly speaking, the limit $N \to \infty$, we are confident that our results hold in arbitrary finite dimension $N \geq 1$. This conviction is built both on physical arguments and on the beautiful results of Carpentier and Le Doussal [4] on the monoscale version of our model in finite dimensions, which, as shown recently, match the exact results of the same model when $N \to \infty$ [5]. The model is defined as follows: the position of the particle, confined inside an N-dimensional spherical box of radius L, is described by the coordinate vector $\mathbf{r} = (r_1, ..., r_N)$, $|\mathbf{r}| \leq L$. It feels a Gaussian-distributed random potential $V(\mathbf{r})$ with zero mean, and with covariance chosen to be isotropic, translation invariant and with a well-defined large N-limit:

$$\langle V(\mathbf{r}_1)\, V(\mathbf{r}_2)\rangle_V = N\, f\left(\frac{1}{2N}(\mathbf{r}_1 - \mathbf{r}_2)^2\right). \qquad (1)$$

In Eq.(1) and henceforth the notation $\langle \dots \rangle_V$ stands for an ensemble average over the random potential, and f is a well behaved function of order unity. The thermodynamics of this model is described by the free energy:

$$F_N = -\beta^{-1} \langle \ln Z(\beta)\rangle_V, \quad Z(\beta) = \int_{|\mathbf{r}|\leq L} \exp -\beta V(\mathbf{r})\, d\mathbf{r} \qquad (2)$$

as a function of the inverse temperature $\beta = 1/T$.

Models of this kind has been studied extensively, and in the high-dimensional limit detailed analytical calculations performed in [6] revealed that the nature of the low temperature phase is essentially dependent on the behavior of the covariance $f(u)$ at large distances. Namely, for short-ranged correlated potentials, the low temperature phase turns out to be described by one-step replica symmetry breaking scheme of Parisi. In contrast, for the case of long-ranged correlated potentials with $f(u)$

growing as $u^{2\gamma}$, the full infinite-hierarchy replica symmetry breaking (FRSB) scheme has to be used.

The problem was reconsidered in much detail recently in [5]. In the limit $N \to \infty$, one actually finds a true phase transition as a function of temperature provided the size of the confining sphere L is scaled as $R\sqrt{N}$, with the parameter $0 < R < \infty$ playing the role of effective radius of the sample. As the existence of a phase transition for finite-size systems is a kind of pathology of the infinite-dimensional approximation, one is mainly interested in the thermodynamic limit $R \to \infty$. A simple analysis then reveals a special role played by *logarithmic* correlation function:

$$f(u) = f_0 - g^2 \ln (u + a^2), \tag{3}$$

where g, a and f_0 are given constants. In particular, only for such case the critical temperature T_c tends to a finite value: $T_c(R \to \infty) = g$. Furthermore, the free-energy found in this limit is given by the well-known Random Energy Model expression [7]. Interestingly, these results coincide *precisely*, up a trivial rescaling, with those obtained earlier for the same potential (3) at $1 \le N < \infty$ dimensions from a Renormalisation Group treatment [4]. In the standard interpretation, below T_c the partition function becomes dominated by a few sites with particularly low random potential, where the particle ends up spending most of its time [8]. Note that the logarithmic growth of the variance of the potential appears naturally in various physical systems of diverse nature, see [4, 9].

The main observation of the present paper is that the above picture, despite looking rather complete, still misses a rich class of possible behavior that survives in the thermodynamic limit $R \to \infty$. Namely, given any increasing positive function $\Phi(y)$ for $0 < y < 1$, we demonstrate below that if one considers potential correlation functions $f(u)$ which take the following scaling form

$$f(u) = -2 \ln R \; \Phi\left(\frac{\ln (u + a^2)}{2 \ln R}\right), \quad 0 \le u < R^2, \tag{4}$$

the thermodynamics of our system in the limit $R \to \infty$ is precisely equivalent to that of celebrated Derrida's GREM [3]. The REM-like case Eq.(3) turns out to be only a (rather marginal) representative of this class: $\Phi(y) = g^2 y$.

Let us explain the motivation of the above form, which will make the physical interpretation of the results quite transparent. The idea is to write $V(\mathbf{r})$ as a (possibly infinite) sum of independent Gaussian potentials: $V(\mathbf{r}) = \sum_{i=1}^{K} V_i(\mathbf{r})$, each with a covariance:

$$\langle V_i(\mathbf{r}_1)\, V_k(\mathbf{r}_2) \rangle_V = \delta_{i,k} N f_i\left(\frac{1}{2N}(\mathbf{r}_1 - \mathbf{r}_2)^2\right), \tag{5}$$

where $f_i(u) = -g_i^2 \ln (u + a^2 + a_i^2)$ as in (3), but each with its own constant g_i, and small-scale cutoffs a_i chosen to grow as a power-law of the system size: $a_i = R^{\nu_i}$ with $0 \le \nu_i \le 1$. Taking the continuum limit $K \to \infty$ with a certain density $\rho(\nu)$ of exponents ν_i, we end up with:

$$f(u) = -\int_0^1 \rho(\nu) g^2(\nu) \ln \left(u + a^2 + R^{2\nu}\right) d\nu, \; 0 \le u \le R^2. \tag{6}$$

Now, introducing $u + a^2 \equiv R^{2y}$ and identifying with Eq. (4) in the $R \to \infty$ limit, we obtain that the function Φ has the following representation:

$$\Phi(y) = y \int_0^y \rho(\nu) g^2(\nu)\, d\nu + \int_y^1 \nu \rho(\nu) g^2(\nu)\, d\nu. \tag{7}$$

Note also that in this representation, $\Phi'(y) = \int_0^y \rho(\nu) g^2(\nu)\, d\nu \ge 0$, and $\Phi''(y) \ge 0$, where the number of dashes here and below indicates the number of derivatives taken. The main result of this work is the following: depending on the nature of the spectrum of the exponents ν, discrete or continuous, we will recover, in the thermodynamic limit, either the free energy of the original GREM with discrete hierarchical structure, or of its continuous hierarchy analogue (see (15) below) analysed recently in much detail by Bovier and Kurkova [10], see also [11].

The physical interpretation of our results is as follows. Instead of one localisation transition temperature T_c at which the particle chooses a finite number of "blobs" of size $O(a)$ where the potential is particularly deep, there appears K different transition temperatures, where the particle localizes on finer and finer length-scales. The largest transition temperature T_1 corresponds a condensation of the Boltzmann weight inside a few blobs of large size $O(R)$, but the particle is still completely delocalized *inside* each blob. As the temperature is reduced, the REM condensation takes place over smaller blobs of size $O(R^\nu)$ inside each already occupied large blobs, and this scenario repeats itself as the temperature is reduced, each time "zooming" in on a smaller scale [12].

The equilibrium free energy per degree of freedom of our model, $F_\infty = \lim_{N\to\infty} F_N/N$ with F_N defined in Eq. (2), can be found in a standard way for any covariance $f(u)$ using the replica trick. The details of the corresponding analysis can be found in [5], and we give below a summary of the most essential formulae for the FRSB situation. For finite R, the low temperature phase is characterised by the existence of a nontrivial, non-decreasing function $x(q)$, $q \in [q_0, q_k]$, with the two parameters q_0 and q_k satisfying the inequality $0 \le q_0 \le q_k \le q_d \equiv R^2$. The corresponding F_∞ can be written in terms of only those two parameters, see Eq.(58) of [5]. Here we choose instead to introduce, along the line of the physical discussion given above, two characteristic "blob" sizes (actually size squared) $d_{\min} = R^2 - q_k$, $d_{\max} = R^2 - q_0$ in terms of which:

$$F_\infty = \frac{1}{2T}\left[f(d_{\min}) - f(0) - d_{\min} f'(d_{\min})\right] - \tag{8}$$
$$-\frac{T}{2}\ln\left[2\pi e d_{\min}\right] + \frac{f'(d_{\max})}{\sqrt{f''(d_{\max})}} - \int_{d_{\min}}^{d_{\max}} \sqrt{f''(u)}\, du,$$

where $d_{\min} \le d_{\max}$ can be found for a given temperature T from the equations

$$0 \le d_{\min} = \frac{T}{\sqrt{f''(d_{\min})}}, \quad d_{\max} = R^2 + \frac{f'(d_{\max})}{f''(d_{\max})} \le R^2. \tag{9}$$

Finally, the Parisi order-parameter function, which takes the values between 0 and 1 and is the main measure of the ultrametricity in the phase space, has the following shape

$$x(d) = -\frac{T}{2}\frac{f'''(d)}{[f''(d)]^{3/2}}, \quad \forall d \in [d_{\min}, d_{\max}]\,. \tag{10}$$

where we performed the overall change $q \to d = R^2 - q$ in comparison with [5]. This function must be now *non-increasing*, and one can verify that this is precisely the case e.g. for the family $f(u)$ in Eq.(4).

The above solution is valid for the temperature range $0 \le T \le T_c$, where the critical temperature T_c is given in terms of the largest blob size $d_{\max}$ as:

$$T_c = d_{\max}\sqrt{f''(d_{\max})}. \tag{11}$$

Above this temperature the solution is replica-symmetric (RS), corresponding to a delocalized phase for the particle: no particular region dominates the partition function. The corresponding free energy is given by:

$$F_\infty = -\frac{T}{2}\ln\left[2\pi d_s\right] + \frac{1}{2T}\left[f(d_s) - f(0)\right] - \frac{T}{2}\frac{R^2}{d_s}, \tag{12}$$

where d_s satisfies

$$d_s = R^2 + \frac{d_s^2}{T^2} f'(d_s)\,. \tag{13}$$

We now consider specifically correlation functions $f(u)$ of the form (4). In what follows we will use the convenient notations $z = (2\ln R)^{-1}$ and $y = z\ln(u + a^2)$. As noted above, our multiscale logarithmic model ensures that $\Phi'(y) \ge 0$ and $\Phi''(y) \ge 0$ for any $0 < y < 1$. We will assume for simplicity $\Phi'(0) = 0$, relegating consideration of the general case to an extended publication [13]. We start our analysis assuming the function $\Phi''(y)$ is finite and differentiable, but later on will relax this condition. Our first goal is to find the largest blob size $d_{\max}$ from second equation in Eq.(9), and then to determine the critical temperature T_c. Introducing the scaling variable $y_{\max} = z\ln(d_{\max} + a^2)$, in the thermodynamic limit $z \to 0$ we can look for a solution $y_{\max}(z)$ as a power series of z. One immediately checks that $y_{\max}(z) = 1 - z\ln 2 + O(z^2)$. This implies that the largest blob size is of the order of the system radius: $d_{\max} \approx R^2/2 \gg a^2$ for $R \to \infty$. Eq.(11) then yields the critical temperature given in the thermodynamic limit by a very simple expression $T_c = \sqrt{\Phi'(1)}$. Physically, at T_c, the sample breaks up into blobs of size $o(R)$ and only a finite number of these blobs are visited by the particle. However, within each blob, all sites are more or less equivalent. Now we can treat along the same lines the first equation in (9) to determine the smallest blob size $d_{\min}$ for $T < T_c$. It can again be conveniently written in terms of the scaling variable $y_{\min} = z\ln(d_{\min} + a^2)$. In the thermodynamic limit $z \to 0$, it is again natural to look for a solution $y_{\min}$ as a power series of z, in which we only retain the first two terms: $y_{\min} = \nu_* + cz + O(z^2)$. Due to our assumption on differentiability of the function $\Phi'(y)$ we expand around $y = \nu_*$, and after a simple calculation find $c = 1$. This means that $d_{\min}$ behaves like $d_{\min} = eR^{2\nu_*}$ for $R \to \infty$, where ν_* satisfies the equation

$$T^2 = \Phi'(\nu_*). \tag{14}$$

Since the function $\Phi'(y)$ is monotonously increasing for $y > 0$, and $\Phi'(0) = 0$ we find that in the limit $R \to \infty$ (i.e. $z \to 0$), the equation Eq.(14) must have a unique solution $0 < \nu^*(T) < 1$ in the range of temperatures $0 < T < T_c = \sqrt{\Phi'(1)}$. In this regime, $d_{\min} \ll d_{\max}$. Physically, sites within blobs of size $d_{\min}$ or smaller are not resolved by the particle, which visits all of them more or less equally.

Now we can easily find the free energy F_∞ by substituting these results to Eq.(8) and extracting the leading term in the thermodynamic limit $z \to 0$. We find the equilibrium free energy to be of the form $F_\infty = (\ln R)\,\mathcal{F}$, where for $0 \le T \le T_c$

$$-\mathcal{F} = T\nu_*(T) + \frac{[\Phi(\nu_*) - \Phi(0)]}{T} + 2\int_{\nu_*}^{1} \sqrt{\Phi'(y)}\, dy. \tag{15}$$

For $T > T_c$ the solution of (13) in the limit $R \to \infty$ is given by $d_s = R^2 \frac{T^2}{T^2 + T_c^2}$ and substituting this to (12) we find that the free energy is given by:

$$-\mathcal{F} = T + \frac{[\Phi(1) - \Phi(0)]}{T}. \tag{16}$$

Last but not least, we can determine the thermodynamic limit of the order-parameter function $x(d)$ given by Eq.(10), which determines in a precise way how the particle localizes on different scales. To leading order in z we find $f''(u) = \Phi'(y)/(u + a^2)^2$, $f'''(u) = -2\Phi'(y)/(u + a^2)^3$, with $y = z \ln(u + a^2)$. Introducing again the scaling variable $\nu = \frac{\ln(d+a^2)}{2\ln R}$ for $d \in [eR^{2\nu_*}, R^2/2]$ we see that the order-parameter function assumes the limiting form:

$$x(\nu) = \frac{T}{[\Phi'(\nu)]^{1/2}}, \quad \forall \nu \in [\nu_*, 1]\,. \tag{17}$$

This completes our solution of the problem for the case of continuous function $\Phi'(y)$. At this point it is rather informative to consider the case of a discrete spectrum of K exponents ν_i, $i = 1, \ldots, K$ satisfying $0 < \nu_K < \nu_{K-1} < \ldots < \nu_1 < \nu_0 = 1$. This corresponds to K superimposed logarithmic potentials with

$$g^2(\nu)\rho(\nu) = \sum_{i=1}^{K} g_i^2\, \delta(\nu - \nu_i), \tag{18}$$

with $\delta(u)$ standing for the Dirac delta-functions. The corresponding $\Phi'(y)$ consists of steps: $\Phi'(y) = \sum_{i=1}^{K} g_i^2 \theta(y - \nu_i)$. A simple consideration shows that our earlier analysis for the values of $d_{\max}$ and the critical temperature T_c still hold for such a case, so $d_{\max} = R^2/2$, and $T_c = [\Phi'(1)]^{1/2} = \sqrt{g_1^2 + g_2^2 + \ldots g_K^2}$. The equation (9) used to determine $d_{\min} = R^{2y_{\min}} - a^2$ now takes the following form:

$$T^2 = \sum_{i=1}^{K} g_i^2 \frac{1 - a^2 e^{-y_{\min}/z}}{1 + e^{(\nu_i - y_{\min})/z}}, \quad z = \frac{1}{2\ln R}. \tag{19}$$

A little thought shows that the solution should always be in the form $y_{\min} = \nu_p + c_p z$ for small z, where the index p runs successively through the values $1, ..., K$ when decreasing temperature from T_c towards $T = 0$. Introducing a decreasing sequence of characteristic temperatures $T_p = \sqrt{\sum_{i=p}^{K} g_i^2}$, we find in the interval $T_{p+1} < T < T_p$ the value $y_{\min} = \nu_p + z \ln(T^2 - T_{p+1}^2)/(T_p^2 - T^2)$. Thus, the value of $y_{\min}$ jumps (and thus the size of the smallest frozen blobs $d_{\min}$) when crossing each of the temperatures T_p, with the highest one being $T_1 = T_c$. It is also clear that for $\nu_p \le \nu < \nu_{p-1}$ one finds: $x(\nu) = T/T_p$ when $T \le T_p$. Since T_p and ν_p decrease as p increases, $x(\nu)$ for a given temperature $T < T_c$ is step-wise constant with jumps at each ν_p; the smaller ν (i.e. the smaller the size of the blobs), the larger $x(\nu)$, meaning that the localisation effect is weaker and finally disappears when $x(\nu) \to 1$.

The expressions for $(y_{\min}, y_{\max})$ suffice to calculate the free energy expression in the thermodynamic limit. In the temperature range $T_{p+1} < T < T_p$ we have

$$-\mathcal{F} = T\nu_p + 2\sum_{i=1}^{p}(\nu_{i-1} - \nu_i)T_i + \frac{1}{T}\sum_{i=p+1}^{K}(\nu_{i-1} - \nu_i)T_i^2, \tag{20}$$

whereas for $T > T_1 = T_c$ the RS expression is

$$-\mathcal{F} = T + \frac{1}{T}\sum_{i=1}^{K}(\nu_{i-1} - \nu_i)\, T_i^2\,. \tag{21}$$

Interestingly, these expressions reproduce exactly, *mutatis mutandis* the free-energy of Derrida's GREM [3, 10], with a particularly clear interpretation in terms of particle localization inside smaller and smaller blobs as the temperature is reduced.

Remembering the mentioned perfect match between the results of [4] and [5] in the limiting case Eq.(3) it is very tempting to conjecture that the GREM behaviour revealed by us in the infinite-dimensional setting should also hold in *all* spatial dimensions, down to $N = 1$, albeit with the largest exponent $\nu_0 < 1$. Indeed, essentially the same mechanisms are at play in both situations. We hope that the corresponding RG and travelling wave formalism of [4] can be generalized to support this conclusion. For finite values of K, where lengthscales are well separated, this looks indeed quite feasible.

If this conjecture is true, we would then have indeed explicitly constructed a Parisi landscape in finite dimensions fully in terms of *stationary* Gaussian processes. How do we reconcile this with the ultrametric properties of the Parisi construction? Consider the following distance D_R defined for any two points $\mathbf{r}, \mathbf{r}'$ inside a sphere of the radius R in the Euclidean space of any dimension:

$$D_R(\mathbf{r}, \mathbf{r}') = \frac{\ln\left[|\mathbf{r} - \mathbf{r}'|^2 + a^2\right]}{2\ln R}, \quad 0 < |\mathbf{r}|, |\mathbf{r}'| \le R. \tag{22}$$

Parameterizing $|\mathbf{r}| \equiv R^{\alpha(\mathbf{r})}$, $0 \le \alpha \le 1$, we see that in fact $\lim_{R\to\infty} D_R(\mathbf{r}, \mathbf{r}') = \max\{\alpha(\mathbf{r}), \alpha(\mathbf{r}')\}$. The latter function used as a distance converts the Euclidean sphere into a so-called ultrametric space: every triangle will have at least two sides equal. We thus conclude that in our model the covariance of the random potential depends only on the ultrametric distance inside our growing sphere, not unlike the original construction of

GREM [3, 10] or directed polymers on a tree with disordered potential [14], cf. discussions in [9, 4] in the single scale case.

Several aspects of the model deserve in our opinion further investigations, some of them to be discussed elsewhere [13]. In particular, the rich behaviour found in the thermodynamics of a single particle should also have interesting dynamical counterparts, cf. [15, 16]. One also can study multifractality exponents reflecting the spatial organization of the Gibbs-Boltzmann weights and the associated singularity spectrum. Finally, let us mention that in $N = 1$ the *mono*scale logarithmic landscape model has in fact deep connections with the *multi*fractal Random Walk construction suggested in [17]. The present model suggests a natural generalisation to a multiscale logarithmic processes [13].

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