

# Collapse of solitary waves near transition from supercritical to subcritical bifurcations

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We study both analytically and numerically the nonlinear stage of the instability of one-dimensional solitons in a small vicinity of the transition point from supercritical to subcritical bifurcations in the framework of the generalized nonlinear Schrödinger equation. It is shown that near the collapsing time the pulse amplitude and its width demonstrate the self-similar behavior with a small asymmetry at the pulse tails due to self-steepening. This theory is applied to both solitary interfacial deep-water waves and envelope water waves with a finite depth and short optical pulses in fibers as well.

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1. According to the usual definition, solitons are nonlinear localized objects propagating uniformly with a constant velocity (see, for example, [1]). The soliton velocity  $V$  represents the main soliton characteristics which defines very often the soliton shape, in particular, its amplitude and width. The velocity  $V$  takes values in the bounded interval with the end points given by the conditions of touching in the  $k$ -space of the plane  $\omega = (\mathbf{k} \cdot \mathbf{V})$  with the dispersion relation of linear waves  $\omega = \omega(\mathbf{k})$ ; intersection of these surfaces means presence of the Cherenkov resonance that prevents the existence of stationary localized entities. At these boundaries the soliton velocity reaches its extremal value  $V_{cr}$  equal to the maximum (or minimum) phase velocity of linear waves and, as a result, solitons there undergo a bifurcation, either supercritical or subcritical. While approaching the supercritical bifurcation point from below or above the soliton amplitude vanishes smoothly according to the same – Landau – law ( $\propto |V - V_{cr}|^{1/2}$ ) as for phase transitions of the second kind. The behavior of solitons in this case is completely universal, both for their amplitudes and their shapes. As  $V \rightarrow V_{cr}$  solitons transform into oscillating wave trains with the carrying frequency corresponding to the extremal phase velocity of linear waves  $V_{cr}$ . The shape of the wave train envelope coincides with that for the soliton of the standard – cubic – nonlinear Schrödinger equation. The soliton width happens to be proportional to  $|V - V_{cr}|^{-1/2}$ .

In the case of subcritical bifurcation, the situation is similar to phase transitions of the first kind: at the

critical velocity the soliton undergoes a jump in its amplitude. In this case the corresponding theory can be developed near the transition point between subcritical and supercritical bifurcations (in analogy with the tricritical point for phase transitions). In the series of papers [2, 3, 4, 5] we demonstrated that in this case the soliton behavior can be described by means of the generalized nonlinear Schrödinger equation (NLSE) for the envelope  $\psi$ , which in the one-dimensional case reads as follows:

$$i \frac{\partial \psi}{\partial t} - \lambda^2 \omega_0 \psi + \frac{\omega_0''}{2} \psi_{xx} - \mu |\psi|^2 \psi + 4i\beta |\psi|^2 \psi_x + \gamma \psi \hat{k} |\psi|^2 + 3C |\psi|^4 \psi = 0, \quad (1)$$

where  $\omega_0 \equiv \omega(k_0)$  and  $k_0$  are the carrying frequency and wave number, respectively,  $\lambda^2 = (V_{cr} - V)/V_{cr} \ll 1$ ,  $\omega_0''$  the second derivative of  $\omega(k)$  taken at  $k = k_0$ . Here the four-wave coupling coefficient  $\mu$  is assumed to have additional smallness characterizing the proximity to the transition from supercritical to subcritical bifurcations. The transition point is defined from the equation  $\mu = 0$ . For example, for interfacial deep-water waves propagating along the interface between two ideal fluids in the presence of capillarity [4, 5]

$$\mu = \frac{k_0^3}{1 + \rho} (A_{cr}^2 - A^2),$$

where  $\rho$  is the density ratio,  $A = (1 - \rho)/(1 + \rho)$  the Atwood number,  $A_{cr}^2 = 5/16$  and  $\rho_{cr} = (21 - 8\sqrt{5})/11$ , as it was shown in Ref. [2]. For  $\rho < \rho_{cr}$ , the four-wave coupling coefficient  $\mu$  is negative, and the corresponding

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nonlinearity is of the focusing type. In this case, solitary waves near the critical velocity  $V_{cr}$  are described by the stationary ( $\partial/\partial t = 0$ ) NLSE and undergo a supercritical bifurcation at  $V = V_{cr}$  [2]. For  $\rho > \rho_{cr}$  the coupling coefficient changes sign and, as a result, the bifurcation becomes subcritical. For water waves in finite depth  $h$  the coefficient  $\mu$  changes its sign at  $\theta_{cr} = k_0 h \approx 1.363$  [6] while  $\omega_0''$  is always negative. Thus the nonlinearity belongs to the focusing type for  $\theta (= kh) > \theta_{cr}$  and respectively becomes defocusing in the region  $\theta < \theta_{cr}$  [6, 7]. In nonlinear optics, as shown in [3], a decrease of  $\mu$  ("Kerr" constant) can be provided by the interaction of light pulses with acoustic waves (Mandelstamm-Brillouin scattering).

Because of the smallness of  $\mu$  we keep in Eq. (1) the next order nonlinear terms: the gradient term ( $\sim \beta$ ) responsible for self-steepening of the pulse (analog of the Lifshitz term in phase transitions), the nonlocal term (due to the presence of the integral operator  $\hat{k}$ , the Fourier transform of its kernel is equal to  $|k|$ ) and the six-wave nonlinear term with coupling coefficient  $C$ . Two additional 4-wave interaction terms, both local and nonlocal, appear as the result of the expansion of the four-wave matrix element  $T_{k_1 k_2 k_3 k_4}$  in powers of the small parameters  $\kappa_i = k_i - k_0$ :

$$T_{k_1 k_2 k_3 k_4} = \frac{\mu}{2\pi} + \frac{\beta}{2\pi}(\kappa_1 + \kappa_2 + \kappa_3 + \kappa_4) - \frac{\gamma}{8\pi}(|\kappa_1 - \kappa_3| + |\kappa_2 - \kappa_3| + |\kappa_2 - \kappa_4| + |\kappa_1 - \kappa_4|). \quad (2)$$

The existence of the nonlocal contribution in the expansion is connected with non-analytical dependence of the matrix element  $T$  in its arguments. For interfacial deep-water waves (IW) this non-analyticity originates from the solution of Laplace equation for the hydrodynamic potential and its reduction to the moving interface. For instance, for water waves (WW) with a finite depth the nonlocal term is absent [7] as well as for electromagnetic waves in nonlinear dielectrics [3] because of the analyticity of matrix elements with respect to frequencies, which is a consequence of causality (see, for example, Refs. [8, 3]). In the latter case the spatial dispersion effects are relativistically small and can be neglected.

For both IW and WW near the transition point,  $\omega_0'' C$  is positive; moreover  $\gamma$  is also positive for IW, and therefore the corresponding nonlinearities are focusing, thus providing the existence of localized solutions. Depending on the sign of  $\mu$  there exist two branches of solitons. For IW they were found numerically [4, 5] using the Petviashvili scheme [9]. Explicit solutions for both kinds of IW solitons can be obtained in the limiting case only when  $V \rightarrow V_{cr}$ . For negative  $\mu$  these are the classical NLS solitons with a sech shape. For the subcritical

bifurcation at  $V = V_{cr}$  the soliton amplitude remains finite with algebraic decay ( $\sim 1/|x|$ ) at infinity [4, 5]. When nonlocal nonlinearity is absent ( $\gamma = 0$ ) soliton solutions can be found explicitly. For both branches at large  $\lambda$  the number of waves  $N = \int |\psi|^2 dx$  approaches from below and above the same value  $N_{cr}$  which coincides with the number of waves  $N$  on the solitons with  $\mu = 0$ . This property for solitons in fibers means that the energy of optical pulse saturates, tending to the constant value with a decrease of the pulse duration.

On the other hand, all solitons of Eq. (1) are stationary points of the Hamiltonian  $H$  for fixed number of waves:  $\delta(H + \lambda^2 N) = 0$ , where the Hamiltonian in dimensionless variables is given by

$$H = \int \left[ |\psi_x|^2 + \frac{\mu}{2} |\psi|^4 + i\beta(\psi_x^* \psi - \psi_x \psi^*) |\psi|^2 - \frac{\gamma}{2} |\psi|^2 \hat{k} |\psi|^2 - C |\psi|^6 \right] dx. \quad (3)$$

This allows one to use the Lyapunov theorem in the analysis of their stability. Here, for the IW,  $\lambda \sim (V_{cr} - V)^{1/2} |\rho - \rho_{cr}|^{-1}$ ,  $C = 319/1281$ ,

$$\mu = \text{sign}(\rho - \rho_{cr}), \quad \beta = 6/\sqrt{427}, \quad \gamma = 32/\sqrt{427}, \quad (4)$$

and for the WW case (compare with Ref. [7])

$$\mu = \text{sign}(\theta_{cr} - \theta), \quad \beta \approx -0.397, \quad C \approx 0.176. \quad (5)$$

As shown in [3, 4, 5, 7], for  $N < N_{cr}$  solitons corresponding to the supercritical branch realize the minimum values of the Hamiltonian and therefore they are stable in the Lyapunov sense, i.e. stable with respect to not only small perturbations but also against finite ones. In particular, the boundedness of  $H$  from below can be viewed if one considers the scaling transformation  $\psi = (1/a)^{1/2} \psi_s(x/a)$  retaining the number of waves  $N$  where  $\psi = \psi_s(x)$  is the soliton solution. Under this transform  $H$  becomes a function of the scaling parameter  $a$ :

$$H(a) = \left( \frac{1}{a} - \frac{1}{2a^2} \right) \frac{\mu}{2} \int |\psi_s|^4 dx. \quad (6)$$

It is worth noting that the dispersion term and all nonlinear terms in  $H$ , except  $\int \frac{\mu}{2} |\psi|^4 dx$ , have the same scaling dependence  $\propto a^{-2}$ . The latter means that at  $\mu = 0$  Eq. (1) can be related to the critical NLS equation like the two-dimensional cubic NLS equation. From Eq. (6) it is also seen that for  $\mu < 0$   $H(a)$  has a minimum corresponding to the soliton. Unlike the supercritical case, the scaling transformation for the other soliton

branch with  $\mu > 0$  gives a maximum of  $H(a)$  on solitons and unboundedness of  $H$  as  $a \rightarrow 0$ . Under the gauge transformation  $\psi = \psi_s e^{i\chi}$ , on the contrary, the Hamiltonian reaches a minimum on soliton solutions and consequently the solitons with  $\mu > 0$  represent saddle points. This indicates a possible instability of solitons for the whole subcritical branch, at least with respect to finite perturbations. In this paper we investigate this question in more details. The main attention will be paid to the nonlinear stage of the instability. This problem, indeed, is not trivial in spite of a closed similarity with the critical NLSE. It is worth noting that Eq. (1) at  $\mu = \gamma = C = 0$  represents an integrable model (the so-called derivative NLSE) [10] and exponentially decaying solitons in this model are stable. It is more or less evident also that small coefficients  $\gamma, C$  cannot break the stability of solitons. This means that in the space of parameters we may expect the existence of a threshold. Above this threshold solitons must be unstable and the development of this instability may lead to collapse, i.e. the formation of a singularity in finite time.

2. Consider the Hamiltonian (3) written in terms of amplitude  $r$  and phase  $\varphi$  ( $\psi = r e^{i\varphi}$ ):

$$H = \int \left[ r_x^2 + \frac{\mu}{2} r^4 - \frac{\gamma}{2} r^2 \widehat{k} r^2 - \frac{1}{3} r^6 + r^2 (\varphi_x + \beta r^2)^2 \right] dx, \quad (7)$$

where by an appropriate choice of the new dimensionless variables the renormalized constant  $\tilde{C} = C + \beta^2$  can be taken equal to  $1/3$ . Hence one can see that the Hamiltonian takes its minimum value when the last term in (7) vanishes, i.e. when

$$\varphi_x + \beta r^2 = 0. \quad (8)$$

Integrating this equation gives an  $x$ -dependence for the phase, called chirp in nonlinear optics. It is interesting to note that the remaining part of the Hamiltonian does not contain the phase at all.

First investigate the local model when  $\gamma = 0$ . Let the Hamiltonian be negative in some region  $\Omega$ :  $H_\Omega < 0$ . Then, following Refs. [11, 12], one can establish that due to radiation of small amplitude waves  $H_\Omega < 0$  can only decrease, becoming more and more negative, but the maximum value of  $|\psi|$ , according to the mean value theorem, can only increase:

$$\max_{x \in \Omega} |\psi|^4 \geq \frac{3|H_\Omega|}{N_\Omega}. \quad (9)$$

This process is possible only for Hamiltonians which are unbounded from below. In accordance with (6) such a

situation is realized when  $\mu > 0$ . In this case the radiation leads to the appearance of infinitely large amplitudes  $r$ . However, it is impossible to conclude that the singularity formation develops in finite time.

For  $\gamma > 0$  the estimations on the maximum value of  $|\psi|$  are not as transparent as they are for the local case. Instead of (9), it is possible to obtain a similar estimate,

$$\max_x |\psi|^4 \geq \frac{3|H|}{N}.$$

However, it is expressed through the total Hamiltonian  $H$  and the total number of waves  $N$ . Besides, two inequalities must be satisfied:  $H < 0$  and  $N < 2N_2/\gamma$ . For interfacial waves,  $N_2 \approx 1.39035 > N_{cr} \approx 1.3521$ . Thus, the maximum amplitude in this case is bounded from below by a conservative quantity and this maximum can never disappear during the nonlinear evolution.

Now we consider the situation where the self-steepening process can be neglected ( $\beta = 0$ ). In this case Eq. (1) becomes

$$i\psi_t + \psi_{xx} - \lambda^2 \psi - \mu |\psi|^2 \psi + \gamma \psi \widehat{k} |\psi|^2 + 3C |\psi|^4 \psi = 0.$$

It is possible to obtain a criterion of collapse using the virial equation (for details, see [13, 11, 14]). This equation is written for the positive definite quantity

$$R = \int x^2 |\psi|^2 dx,$$

which, up to the multiplier  $N$ , coincides with the mean square size of the distribution. The second derivative of  $R$  with respect to time is defined by the virial equation

$$R_{tt} = 8 \left( H - \frac{\mu}{4} \int |\psi|^4 dx \right). \quad (10)$$

Hence, for  $\mu > 0$  one can easily obtain the following inequality:

$$R_{tt} < 8H,$$

which yields, after double integration,  $R < 4Ht^2 + \alpha_1 t + \alpha_2$ . Here  $\alpha_{1,2}$  are constants which are obtained from the initial conditions. Hence, it follows that for the states with negative Hamiltonian,  $H < 0$ , there always exists such moment of time  $t_0$  when the positive definite quantity  $R$  vanishes. At this moment of time the amplitude becomes infinite. Therefore the condition  $H < 0$  represents a sufficient criterion of collapse (compare with [13, 11]). However, it is necessary to add that this criterion can be improved by the same way as it was done in Refs. [15, 16] for the three-dimensional cubic NLS equation. From Eq. (10) one can see that for the stationary case (on the soliton solution)  $H_s = \frac{\mu}{4} \int |\psi_s|^4 dx$ ,

in agreement with Eq. (6). As we demonstrated before for  $\mu > 0$  the soliton realizes a saddle point of  $H$  for fixed  $N$ . It follows from (6) that for small  $a$  the Hamiltonian becomes unbounded from below, but for  $a > 1$  it decreases (this corresponds to spreading). Therefore in order to achieve a blow-up regime the system should pass through the energetic barrier equal to  $H_s$ . Thus, for this case the criterion  $H < 0$  must be changed into the sharper criterion:  $H < H_s$ . This criterion can be obtained rigorously using step by step the scheme presented in [15, 16] and therefore we skip its derivation.

**3.** In order to verify all the theoretical arguments about the formation of collapse presented above we performed a numerical integration of the NLSE (1) for  $\mu > 0$  by using the standard 4th order Runge-Kutta scheme. The initial conditions were chosen in the form of solitons but with larger amplitudes than for the stationary solitons. The increase in initial amplitude was varied in the interval from 0.1% up to 10%. The initial phase was given by means of Eq. (8). In all runs with these initial conditions we observed a high increase of the soliton amplitude up to a factor 14 with a shrinking of its width. In the peak region pulses for both IW and WW cases behaved similarly. Near the maximum the pulse peak was almost symmetric: anisotropy was not visible. The difference was observed in the asymptotic regions far from the pulse core where the pulses had different asymmetries for IW and WW because of the opposite sign for  $\beta$  (see Eqs. (4), (5)). For the given values of  $\beta$  we did not observe the simultaneous formation of two types of singularities with blowing-up amplitudes and sharp gradients as it was demonstrated in the recent numerical experiments for the three-dimensional collapse of short optical pulses due to self-focusing and self-steepening in the framework of the generalized NLS equation [18] and equations of the Kadomtsev-Petviashvili type [19].

In our numerical computations we found that the amplitude and its spatial collapsing distribution develop in a self-similar manner. Near the collapse point in the equation (with  $\mu > 0$ ) one can neglect the term proportional to  $\mu$ . In this asymptotic regime Eq. (1) admits self-similar solutions,

$$r(x, t) = (t_0 - t)^{-1/4} f\left(\frac{x}{(t_0 - t)^{1/2}}\right), \quad (11)$$

where  $t_0$  is the collapse time.

To verify that we approached the asymptotic behavior given by Eq. (11), we normalized at each moment of time the  $\psi$ -function by the maximum (in  $x$ ) of its modulus  $\max |\psi| \equiv M$  and introduced new self similar variables (compare with [17]),

$$\psi(x, t) = M\bar{\psi}(\xi, \tau), \quad \xi = M^2(x - x_{\max}), \quad \tau = \ln M. \quad (12)$$

Here  $x_{\max}$  is the point corresponding to the maximum of  $|\psi|$ . In comparison with those given by Eq. (11), such new variables are more convenient because they do not require the determination of the collapsing time  $t_0$ .

Fig.1 and Fig.2 show typical dependences of  $|\bar{\psi}|$  as a function of the self-similar variable  $\xi$  at  $t = 0$  (solid

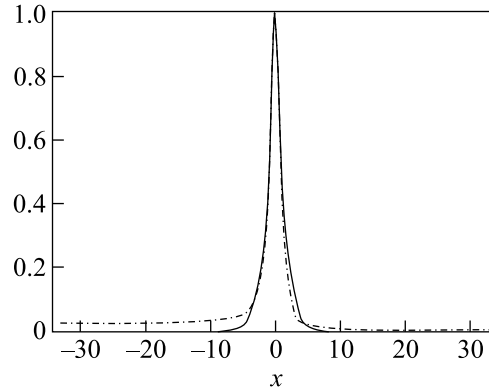


Fig.1. Initial (solid line) and final (dashed line) at  $t = 1.18$  distributions for  $|\bar{\psi}|$ , interfacial waves, self-similar variables. The soliton amplitude was increased by 1%,  $\mu = 1$ ,  $\lambda = 1$ . The ratio between final and initial soliton amplitudes in the physical variables is about 11

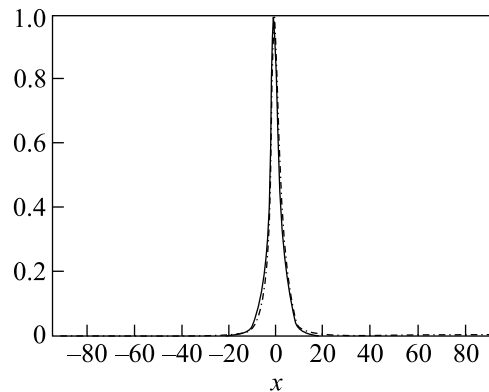


Fig.2. Initial (solid line) and final (dashed line) at  $t = 2.7192$  distributions for  $|\bar{\psi}|$ , WW solitons, self-similar variables. The soliton amplitude was increased by 1%,  $\mu = 1$ ,  $\lambda = 1$ . The ratio between final and initial soliton amplitudes in the physical variables is about 11

line) and at the final time (dashed line) for both the IW and WW cases. In both figures one can see a fairly good coincidence between the initial soliton distribution and the final one at the central (collapsing) part of the pulse and asymmetry of the pulse at its tails due to self-steepening. The latter demonstrates that collapse has a

self-similar behavior. The form of the central part of the pulse approaches the soliton shape because asymptotically the NLS model (1) tends to the critical NLS system. It is necessary to mention that this has been well-known for the classical two-dimensional NLS equation since the paper by Frailman [20].

Fig.3 shows how  $1/\max|\psi|^4$  depends on time. This dependence is almost linear in the correspondence with

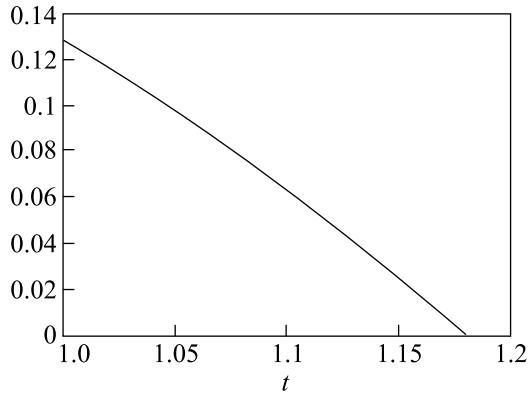


Fig.3. Dependence of  $1/\max|\psi|^4$  on time. Interfacial waves

the self-similar law (11). If the initial amplitudes were less than the stationary soliton values, then the distribution would spread in time dispersively, in full correspondence with qualitative arguments based on the scaling transformations (6).

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