Pis'ma v ZhETF, vol. 88, iss. 1, pp. 3 – 6

Reaction $e^+e^- \to \pi^+\pi^-\pi^+\pi^-$ at energies $\sqrt{s} \leq 1$ GeV

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Submitted 14 May 2008

The cross section of reaction $e^+e^- \to \pi^+\pi^-\pi^+\pi^-$ is calculated for energies $0.65 \leq \sqrt{s} \leq 1$ GeV in the framework of the generalized hidden local symmetry model. The calculations are compared with the data of CMD-2 and BaBaR. It is shown that the inclusion of heavy isovector resonances $\rho(1450)$ and $\rho(1700)$ is necessary for reconciling calculations with the data. It is found that at $\sqrt{s} \approx 1$ GeV the contributions of above resonances are much larger, by the factor of 30, than the $\rho(770)$ one, and are amount to a considerable fraction $\sim 0.3{-}0.6$ of the latter at $\sqrt{s} \sim m_\rho$.

PACS: 11.30.Rd, 12.39.Fe, 13.30.Eg

Among chiral models aimed at the description of interactions of the pseudoscalar mesons with the low lying vector and axial vector ones, see the review [1] and references therein, the most elegant is the generalized hidden local symmetry (GHLS) model [2]. It relates all coupling constants to only the pion decay constant f_π and $g_{\rho\pi\pi}$, and accounts for anomalous processes in a way that does not break low energy theorems. Strikingly, but this very popular model was not scrutinized in the processes with sufficiently soft pions where one can rely on the tree approximation. The purpose of the present paper is to fill this gap by plotting the $e^+e^- \to \pi^+\pi^-\pi^+\pi^-$ reaction cross section in the GHLS model and comparing the results with available data CMD-2 [3] and BaBaR [4]. When so doing, we use our recent calculations of the $\rho \to 4\pi$ decay amplitudes [5, 6] to account for the resonant production $e^+e^- \to \rho \to \pi^+\pi^-\pi^+\pi^-$. Note that excitations curves in [5] do not include the $a_1\pi$ intermediate state [6] nor the contact non-resonant contributions $e^+e^- \to \gamma^* \to \rho\pi\pi \to 4\pi$, $e^+e^- \to \gamma^* \to a_1\pi \to 4\pi$ whose explicit form is found here.

The ingredients for the amplitude with the resonant ρ meson are given in [5, 6]. The Lagrangian of the direct photon coupling is

$$\mathcal{L}_{\rm photon} = -e\mathcal{A}_\mu\left(2gf_\pi^2\rho^0_\mu - \frac{\pi^+\pi^-}{2f_\pi^2}[\boldsymbol{\pi}\times\partial_\mu\boldsymbol{\pi}]_3 - - 2g\rho^0_\mu\pi^+\pi^- + 2gf_\pi[\boldsymbol{\pi}\times\mathbf{a}_\mu]_3\right), \qquad (1)$$

where $g = g_{\rho\pi\pi}$, and $\mathcal{A}_\mu$, $\mathbf{a}_\mu$, $\boldsymbol{\pi}$ stand for the photon four-vector potential, $a_1(1260)$, π meson field, respectively. Boldface characters refer to isotopic vectors. Given are only the terms necessary for the $\pi^+\pi^-\pi^+\pi^-$ final state, and the contributions of the second order in electric charge e are neglected. Note that the contact $\gamma^* \to \pi^+\pi^-$ and $\gamma^* \to \pi^+\pi^-\pi^+\pi^-$ vertices cannot be simultaneously eliminated in HLS, while the contact $\gamma^* \to \pi^+\pi^-$ vertex is eliminated in HLS by the parameter choice [2].

It is suitable to represent the energy dependence of the $e^+e^- \to \pi^+\pi^-\pi^+\pi^-$ reaction cross section in the form

$$\sigma_{e^+e^-\to4\pi}(s) = \frac{12\pi m_\rho^3\Gamma_{\rho e^+e^-}(m_\rho)\Gamma^{\rm eff}_{\rho\to4\pi}(s)}{s^{3/2}|D_\rho(q)|^2}, \qquad (2)$$

where the leptonic width of the vector meson V on the mass shell looks as

$$\Gamma_{Ve^+e^-}(m_V) = \frac{4\pi\alpha^2 m_V}{3f_V^2}, \qquad (3)$$

and $s = q^2$ is the total energy squared in the center-of-mass system. The function $\Gamma^{\rm eff}_{\rho\to4\pi}(s)$ in (2) is evaluated with the effective $\rho \to 4\pi$ decay amplitude $M^{\rm eff}_{\rho\to4\pi} \equiv \equiv M^{\rm eff}_{\rho_q\to\pi^+_{q_1}\pi^+_{q_2}\pi^-_{q_3}\pi^-_{q_4}}$ which includes both the resonant contribution $e^+e^- \to \gamma^* \to \rho \to \pi^+\pi^-\pi^+\pi^-$ and the contact one $e^+e^- \to \gamma^* \to \pi^+\pi^-\pi^+\pi^-$. In the lowest order in electromagnetic coupling constant this amplitude is given by the expression

$$M^{\rm eff}_{\rho\to4\pi} = \frac{g_{\rho\pi\pi}}{f_\pi^2}\epsilon_\mu(A_1q_{1\mu} + A_2q_{2\mu} + A_3q_{3\mu} + A_4q_{4\mu}), \qquad (4)$$

where ϵ_μ stands for the polarization four-vector of the virtual ρ meson, and $A_a \equiv A_a(q_1, q_2, q_3, q_4)$, $a = = 1, 2, 3, 4$ are dimensionless invariant functions. $A_1 = = -1 + (1 + \widehat{P}_{34})B_1$, where

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$$
\begin{aligned}
B_1 = {}& \frac{2}{D_\pi(q-q_1)}\left[\frac{m_\rho^2}{D_{\rho 23}}(q_4, q_2-q_3)-(q_2,q_3)\right] - \\
& - D_\rho(q)\left(\frac{1}{D_{\rho 14}}-\frac{1}{2m_\rho^2}\right) - \frac{(1-\widehat{P}_{23})}{4D_{a_1}(q-q_1)} \times \\
& \times \left\{\frac{1}{D_{\rho 23}}\left[4(q_2,q_4)(2q-q_1,q_3) - \right.\right. \\
& - \left.(2q-q_1,q_2)(q_4,q-q_1)+(2q-q_1,q_4)(q_2,q_4)\right] - \\
& - \frac{1}{2m_\rho^2}(q_2, 2q-2q_1+q_4)(2q-q_1,q_3) - \\
& - \frac{4(q_2,q_4)(q_2+q_3)^2}{m_{a_1}^2}\left[q^2+D_\rho(q)\right]\left(\frac{1}{D_{\rho 23}} - \right. \\
& - \left.\left.\frac{1}{8m_\rho^2}\right)\right\} - \frac{3(q,q_2)-m_\pi^2-D_\rho(q)}{4D_{a_1}(q-q_2)} \times \\
& \times \left[\frac{(q_4, 4q_3-q_2+q)}{D_{\rho 13}} - \frac{(q_4, 2q-2q_2+q_3)}{2m_\rho^2}\right] + \\
& + \frac{(q_4,q_2-q_3)}{4D_{\rho 23}} + \frac{(q_4,q_2+q_3)}{4m_\rho^2} - \\
& - \frac{3(q,q_4)-m_\pi^2-D_\rho(q)}{4D_{a_1}(q-q_3)}\left[\frac{(q_2, 4q_3-q_4+q)}{D_{\rho 13}} - \right. \\
& - \left.\frac{(q_1,q_2-q_3)}{D_{\rho 23}} - \frac{(q_3, 2q-2q_4+q_2)}{2m_\rho^2}\right] - \frac{(q_2,q_3)}{2D_{\rho 14}}. \quad (5)
\end{aligned}
$$

The notations are: $\widehat{P}_{ab}$ is the operator interchanging the pion momenta $q_a \leftrightarrow q_b$, $D_{\rho ab} \equiv D_\rho(q_a+q_b)$ is the inverse propagator of ρ meson with the invariant mass squared $(q_a+q_b)^2$,

$$D_\rho(q) = m_\rho^2 - q^2 - i\sqrt{q^2}\Gamma_\rho(\sqrt{q^2}), \quad (6)$$

see (3.3)−(3.5) in [6] for $\Gamma_\rho(\sqrt{q^2})$. The terms $\propto D_\rho(q)$ in (5) refer to the contact terms generated by (1). (P,Q) stands for invariant scalar product of two four-vectors P and Q, $D_\pi(p) = m_\pi^2 - p^2$ is the inverse propagator of pion, m_π and m_ρ are the masses of charged pion and $\rho(770)$ meson taken from [7]. A_2 is obtained from A_1 by interchanging $q_1 \leftrightarrow q_2$, A_3 is obtained from A_1 by simultaneous interchanges $q_1 \leftrightarrow q_3$, $q_2 \leftrightarrow q_4$ followed by inverting an overall sign, and A_4 is obtained from A_3 by interchanging $q_3 \leftrightarrow q_4$. The form of the a_1 propagator $D_{a_1}^{-1}$ with the energy dependent width is given in [6]. Here $\Gamma_{a_1} \neq 0$ should be taken into account because $\sqrt{s} = 1$ GeV is close to $m_{a_1} = 1.23$ GeV (a PDG value [7]) or to $m_{a_1} = \sqrt{2}m_\rho = 1.09$ GeV given by Weinberg's relation. We use the approximate expression for $\Gamma_{a_1}(m)$ which interpolates the curve in [6] in the range $3m_\pi \le m \le \sqrt{s} - m_\pi$, $\sqrt{s} \le 1$ GeV.

The resonant contribution $\gamma^* \to \rho \to \pi^+\pi^-\pi^+\pi^-$ in (4) respects the requirement of chiral symmetry in that it vanishes at the vanishing momentum $q_{a\mu} \to 0$ ($a = 1$, 2, 3, 4) of any final pion, provided $m_\pi = 0$. However, the terms due to the direct $\gamma^* \to \pi^+\pi^-\pi^+\pi^-$ contribution do not vanish in the above limiting cases. This is the consequence of the breaking of conservation of the axial current by electromagnetic field, $\partial_\mu j^a_{\mu,A} = e\mathcal{A}_\mu \epsilon_{3ab} j^b_{\mu,A}$ upon neglecting the term $\propto m_\pi^2$. One can show that the terms in (4) surviving in the limit $q_{a\mu} \to 0$, correspond to the matrix elements of the above divergence of axial current.

The results of evaluation of the $e^+e^- \to \pi^+\pi^-\pi^+\pi^-$ reaction cross section in GHLS model are shown in Fig.1. The curves are obtained in the case $m_{a_1} =$ $= 1.23$ GeV; the results for the mass $m_{a_1} = 1.09$ GeV look qualitatively the same. One can see that the model is unable to reproduce the magnitude of the cross section at energies $\sqrt{s} > 0.8$ GeV. Let us include the contributions of heavier resonances $\rho' \equiv \rho(1450)$ and $\rho'' \equiv \rho(1700)$ trying to explain the cross section magnitude at $\sqrt{s} \ge 0.8$ GeV, without invoking the higher derivative terms in the effective lagrangian. We choose the simplest parametrization consisting of the Breit-Wigner resonance shape with the constant widths and masses $m_{\rho'} = 1.459$ GeV, $\Gamma_{\rho'} = 0.147$ GeV, $m_{\rho''} =$ $= 1.72$ GeV, $\Gamma_{\rho''} = 0.25$ GeV taken from [7] and neglect the $\rho(770) - \rho(1450) - \rho(1700)$ mixing due to their common decay modes. This approximation results in no qualitative difference in the role of heavy resonance at $\sqrt{s} \le 1$ GeV as compared to more sophisticated models with mixing. We also adopt the assumption of $a_1\pi$ dominance in the $\rho', \rho'' \to 4\pi$ decay dynamics [8], but modify

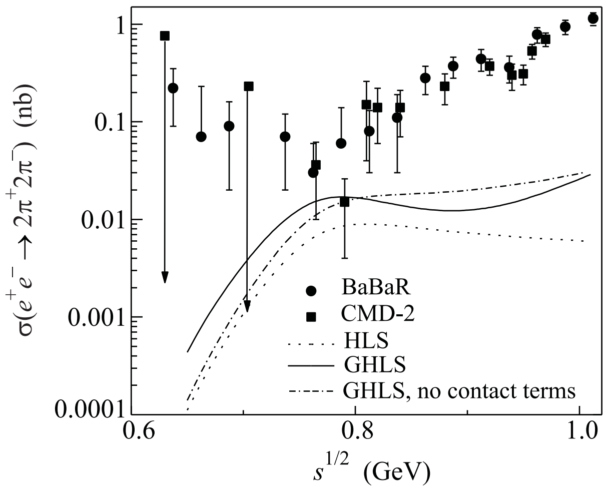

Fig.1. The energy dependence of the $e^+e^- \to \pi^+\pi^-\pi^+\pi^-$ reaction cross section in the generalized hidden local symmetry model; $m_{a_1} = 1.23$ GeV. The data are CMD-2 [3] and BaBaR [4]. "HLS" refers to the case of no a_1, no contact terms, "GHLS" does to one with both a_1 meson and contact couplings (1). "GHLS, no contact terms" refers to the model without contact terms

it to include the requirements of chiral symmetry. Then taking into account the ρ', ρ'' resonance contributions results in the factor

$$R(s) = \left|1 + \frac{D_\rho(q)}{1+r(s)}\left[\frac{x_{\rho'}}{D_{\rho'}(q)} + \frac{x_{\rho''}}{D_{\rho''}(q)}\right]\right|^2, \qquad (7)$$

multiplying the right hand side of (2), where $D_V(q) = m_V^2 - s - im_V\Gamma_V$, $V = \rho', \rho''$, $s = q^2$. Free parameters $x_{\rho'}$ and $x_{\rho''}$ are found from fitting the data. The meaning of $x_{\rho'}$ is that

$$x_{\rho'} = \frac{g_{\gamma\rho'}}{g_{\gamma\rho}}\frac{g_{\rho'\to a_1\pi\to 4\pi}}{g_{\rho\to a_1\pi\to 4\pi}}, \qquad (8)$$

analogously for $x_{\rho''}$, where $g_{\gamma V} = em_V^2/f_V$ is the photon-vector meson V transition amplitude, f_V is related with the leptonic width (3). Since ρ and ρ' are assumed here to have the similar coupling to the state $a_1\pi$, the ratio (8) is constant. The complex function $r(s)$ in (7) is the ratio of the amplitude with the intermediate a_1 meson to one with no a_1 contribution. It approximately takes into account the $a_1\pi$ dominance in the four pion decay of heavy isovector resonances and is precalculated for the CMD-2 [3] and BaBaR [4] data points $\sqrt{s} \le 1$ GeV:

$$r(s) = \left[\frac{\Gamma^{\rm eff,no a_1}_{\rho\to 4\pi}}{\Gamma_{\rho\to a_1\pi\to 4\pi}}\right]^{1/2}\exp(i\chi),$$

$$\chi = \cos^{-1}\frac{\Gamma^{\rm eff}_{\rho\to 4\pi} - \Gamma^{\rm eff,no a_1}_{\rho\to 4\pi} - \Gamma_{\rho\to a_1\pi\to 4\pi}}{2\sqrt{\Gamma_{\rho\to a_1\pi\to 4\pi}\Gamma^{\rm eff,no a_1}_{\rho\to 4\pi}}}. \qquad (9)$$

Here $\Gamma_{\rho\to a_1\pi\to 4\pi} \equiv \Gamma_{\rho\to a_1\pi\to 4\pi}(s)$ is the $\rho^0 \to \pi^+\pi^-\pi^+\pi^-$ decay width due to the intermediate $a_1\pi$ state only, while $\Gamma^{\rm eff,no a_1}_{\rho\to 4\pi} \equiv \Gamma^{\rm eff,no a_1}_{\rho\to 4\pi}(s)$ is the effective width of the same decay including all the contribution mentioned above except the $a_1\pi$ one. The approximation (9) corresponds to the averaging over four pion phase space necessary to evade unacceptably long time in the fitting procedure.

Table 1

The results of fitting CMD-2 data [3]

	$x_{\rho'}$	$x_{\rho''}$	$\chi^2/N_{\rm d.o.f}$	m_{a_1} [GeV]
1	-27.5 ± 1.5	$\equiv 0$	15.4/10	1.23
2	$\equiv 0$	-46.2 ± 2.5	15.4/10	1.23
3	96.8 ± 1.5	-208.7 ± 2.5	14.5/9	1.23
4	-17.8 ± 1.0	$\equiv 0$	15.7/10	1.09
5	$\equiv 0$	-30.1 ± 1.5	15.4/10	1.09
6	72.5 ± 1.0	-151.9 ± 1.6	14.7/9	1.09

The results of fitting the CMD-2 data are given in Table 1. The curves corresponding to the fit variant 3 with ρ' and ρ'' resonances are shown in Fig.2. This

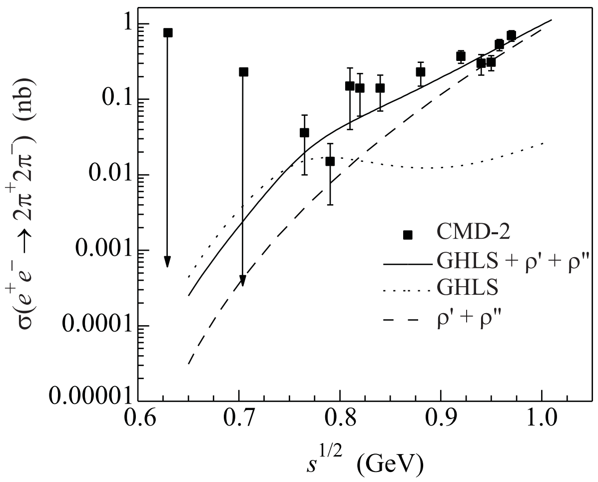

Fig.2. The results of fitting the CMD-2 data [3]. “GHLS” refers to the model without ρ', ρ''

variant is indistinguishable from the variants with the single ρ' (variant 1) or ρ'' (variant 2), both resulting in the same curves as the dashed one shown in Fig.2. Variants 4 – 6 correspond to the fits with the mass $m_{a_1} = m_\rho\sqrt{2} = 1.09$ GeV and result in the same corresponding curves not shown here. The quality of fit is not quite good. Nevertheless, we quote the contribution of the sum $\rho' + \rho''$ (variant 3) or ρ' (variant 1) and ρ'' (variant 2) relative to the case of pure GHLS contribution (dotted line in Fig.2) to be 0.3 at $\sqrt{s} \approx m_\rho$ and 32 at $\sqrt{s} = 1$ GeV. These numbers refer to the case $m_{a_1} = 1.23$ GeV. The case $m_{a_1} = 1.09$ GeV results in almost the same figures for above ratios.

Table 2

The results of fitting BaBaR data [4]

	$x_{\rho'}$	$x_{\rho''}$	$\chi^2/N_{\rm d.o.f}$	m_{a_1} [GeV]
1	-25.2 ± 0.9	$\equiv 0$	32.6/16	1.23
2	$\equiv 0$	-44.0 ± 2.1	29.3/16	1.23
3	273.2 ± 1.4	-514.5 ± 2.3	11.2/15	1.23
4	-15.8 ± 0.8	$\equiv 0$	35.0/16	1.09
5	$\equiv 0$	-27.7 ± 1.3	31.8/16	1.09
6	198.5 ± 1.0	-370.1 ± 1.5	11.2/15	1.09

The results of the similar analysis of the BaBaR data [4] are presented in Table 2. Contrary to the previous case, here the variants with the single additional heavy resonance give a bad description. The fit chooses two destructively interfering ρ' and ρ'' resonances each coupled to $a_1\pi$ much strongly than in the variants of the single heavy resonance. The curves shown in Fig.3 refer

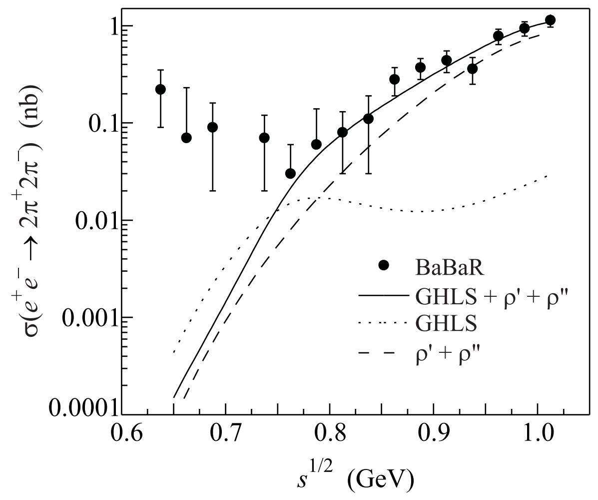

Fig.3. The same as in Fig.2, but for the BaBaR data [4]

to variant 3 in Table 2 with $m_{a_1} = 1.23\,\mathrm{GeV}$. The contribution of the sum $\rho' + \rho''$ (in variant 3) or ρ' (variant 1) and ρ'' (variant 2) relative to the case of pure GHLS contribution (dotted line in Fig.3)is found to be 0.6 at $\sqrt{s} \approx m_\rho$ and 30 at $\sqrt{s} = 1\,\mathrm{GeV}$. As in the case of the CMD-2 data, here the variant 6 with $m_{a_1} = 1.09\,\mathrm{GeV}$ results in practically the same corresponding curves and ratios.

Our conclusions differ from the result of the works [3, 9, 10] all claiming small or even absent contribution of heavy resonances. We attribute this disagreement to the difference among the models used in the present analysis and in works [3, 8–10]. The works [3, 10] exploit non-chiral invariant effective Lagrangians. The work [9] is based on chiral amplitude with three unknown parameters. No central values nor their errors are given in order to assess independently the quality of approach [9]. The effective vertex $a_1\rho\pi$ used in that work refers to the higher derivative contribution, while there exists a lowest derivative one used in the present work, see [6]. The contact $\gamma\pi^+\pi^-$ vertex is present in the intermediate state of the amplitude in [9]. The apparent violation of the vector dominance of the pion form factor could be evaded by adjusting arbitrary constants be in [9] only assuming the vanishing of the ρ meson width which is inappropriate in the energy range where the ρ width is essential.

Thus, the simplest variant of GHLS model with the minimal number of derivatives fails to explain the cross section of the reaction $e^+e^- \to \pi^+\pi^-\pi^+\pi^-$ at energies $0.8 < \sqrt{s} \leq 1\,\mathrm{GeV}$. One possible way out this difficulty by including heavy resonances ρ', ρ'' is studied here. GHLS model is based on the nonlinear realization of chiral symmetry. It would be desirable to readdress the present issues in the frame work of the chiral model of the vector and axial vector mesons based on the linear σ-model. This task is necessary in order to evaluate the robustness of the figures characterizing the contributions of heavier resonances towards various model assumptions and to reveal the role of the intermediate states which include the widely discussed scalar σ meson.

The work is partially supported by the grants of the Russian Foundation for Basic Research # RFBR-07-02-00093 and of the Support of the Leading Scientific Schools # Sh-5362.2006.2.

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