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Vacuum energy, cosmological constant and Standard Model physics

G. L. Alberghi[+], *A. Yu. Kamenshchik*[+*1)], *A. Tronconi*[+], *G. P. Vacca*[+], *G. Venturi*[+]

[+] *Dipartimento di Fisica and INFN, 40126 Bologna, Italy*

[*] *L.D. Landau Institute for Theoretical Physics RAS, 117940 Moscow, Russia*

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The conditions for the cancellation of one loop contributions to vacuum energy (both U.V. divergent and finite) coming from the Standard Model (SM) fields are examined. It is proven that this is not possible unless one introduces besides several bosons, at least one massive fermion having mass within specific ranges. On examining one of the simplest SM extensions satisfying the constraints one finds that the mass range of the lightest massive boson is compatible with the present Higgs mass bounds. We briefly address the possible treatment of the presence of interactions.

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1. Introduction. Almost sixty years ago Pauli [1, 2] suggested that the vacuum (zero-point) energies of all existing fermions and bosons compensate each other. This possibility is based on the fact that the vacuum energy of fermions has a negative sign whereas that of bosons has a positive one. We note that such an idea is realized in a highly constrained way in supersymmetric models, although supersymmetry breaking must be present at probed energies in order to explain observed data. Subsequently in a series of papers Zeldovich [3] connected the vacuum energy to the cosmological constant, however rather than eliminating the divergences through a boson-fermion cancellation he suggested a Pauli-Villars regularization of all divergences introducing a spectrum of massive regulator fields. Covariant regularization of all contributions then leads to finite values for both the energy density and (negative) pressure corresponding to a cosmological constant, i.e. connected by the equation of state $p = -\varepsilon$, where p is the pressure, while ε is the energy density.

The approach, which we develop in this paper is based on the combination of these two ideas: as Pauli suggests the *existing* boson and fermion fields provide the exact cancellation of all the ultraviolet divergences in the vacuum energy, while as Zeldovich suggests, the remaining finite part of the vacuum energy leads to an effective cosmological constant. Thus, rather than use a regularization approach we shall assume that the actual particle content of a theory is such that U.V. divergences do not appear insofar bosons and fermions should compensate each other as Pauli suggested. Indeed we have previously examined [4] the problem of U.V. divergences of the vacuum energy for both Minkowski and de Sitter space-times and formulated the conditions for the cancellation of all divergences. These conditions lead to strong restrictions on the spectra of possible elementary particle models. In this note we shall apply such considerations to the observed particles of the Standart Model (SM) and also study the finite part of the vacuum energy and the possibility of a cancellation for this contribution also, so as to obtain a result compatible with the observed value of the cosmological constant (almost zero with respect to SM particle masses). The cancellation of all one-loop contributions to the cosmological and Newton constants was also considered in the context of the induced gravity approach [5]. We shall instead consider Einstein gravity and obtain all constraints in Minkowski space which is implicit if all contributions (divergent and finite) compensate between fermions and bosons. Let us illustrate them in order.

The requirement that quartic divergences cancel is just that the numbers of bosonic and fermionic degrees of freedom be equal ($N_B = N_F$). The conditions for the cancellation of quadratic and logarithmic divergences on a flat Minkowski background are

$$\sum m_s^2 + 3\sum m_V^2 = 2\sum m_F^2 \qquad (1)$$

and

$$\sum m_s^4 + 3\sum m_V^4 = 2\sum m_F^4\,, \qquad (2)$$

respectively. Here the subscripts s, V and F denote scalar, massive vector and massive spinor Majorana fields respectively (for Dirac fields it is enough to put 4 instead of 2 on the right-hand sides of Eqs. (1) and (2)). For the case of a de Sitter spacetime, equations giving conditions for the cancellation of quadratic and logarithmic ultraviolet divergences are more involved.

[1)] e-mail: kamenshchik@bo.infn.it

Some examples of these conditions for simple particle physics models have been presented in Ref. [4].

As was shown in Ref. [3] the finite part of the vacuum energy density is equal to

$$\varepsilon = \frac{1}{8}\int d\mu f(\mu)\mu^4 ln\mu, \tag{3}$$

where $f(\mu)$ is the spectral function. The requirement that the finite part of the vacuum energy (and pressure) is very small compared with SM masses suggests that we also need a compensation between the finite parts of fermion and boson vacuum energies, obtaining

$$\sum m_s^4 \ln m_s + 3\sum m_V^4 \ln m_V - 2\sum m_F^4 \ln m_F = 0. \tag{4}$$

This leads to a zero cosmological constant (Minkowski space).

As is known the observed number of fermionic degrees of freedom in the SM is much higher than the number of bosonic degrees of freedom [6]. Indeed N_F is equal to 96 (if we consider the neutrinos as massive particles) while the number of bosonic degrees of freedom, carried by the photon, the gluons and the $W^{\pm}$ and Z^0 bosons is equal to 27. Thus we need an additional 69 boson degrees of freedom, one of which is the Higgs boson. Ideally we would like to obtain some minimal extension of the SM, which would not modify the fermionic degrees of freedom while just adding hypothetical bosons.

Our main result is a proof that, within the given framework, such an extension does not exist. In other words, we show that on introducing new bosonic fields, which provide the cancellation of the ultraviolet divergences in the vacuum energy density, the finite part of the effective cosmological constant is always positive and of order of the mass of the top quark to the fourth power, which is much higher than the value of the effective cosmological constant compatible with the cosmological observations. This leads to the necessity of introducing new heavy fermions. Indeed we shall find explicit realizations with zero finite energy once one introduces at least one fermion with a suitable mass.

Following a general analysis, for the sake of simplicity we shall consider an explicit minimal extension of the SM with a few massive bosons and weakly coupled, practically massless, others so as to satisfy the requirement $N_B = N_F$. Such a possibility is viable in effective action approaches and, for example, has been considered recently in such scenarios as unparticle physics [7]. In this minimal framework we shall analyze the boson masses allowed by the cancellation constraints.

It is obvious that one may study vacuum energy in the more modern and general setting [8] of effective actions, renormalization group flows, and even attempt to include the effects of condensates, however we feel that it is worthwhile to first examine the full consequences for SM physics of Pauli's original suggestion.

In next section we study the consequences of our equations for the SM particle spectrum and in the last section our results are summarized and discussed.

2. The Standard Model and vacuum energy balance. Let us begin by observing that the mass of the top quark $m_t \approx 170\,\mathrm{GeV}$ is much higher than the masses of all other fermions (the bottom quark has the mass $m_b \approx 4.5\,\mathrm{GeV}$ while the mass of the heaviest τ-lepton is $m_\tau \approx 2\,\mathrm{GeV}$). Thus, on considering Eqs. (1), (2) and (4) we can limit ourselves to only taking into account the contributions of the top quark, whose mass is conveniently used as the reference unit mass, and of the massive vector bosons. Then the mass of $W^{\pm}$ bosons is $m_W \approx 0.47 m_t$ while that of the Z^0 boson is $m_Z \approx 0.53 m_t$, with $m_t = 1$.

Quantities describing the contributions of the heavy fermion and boson degrees of freedom in the conditions (1), (2) and (4) are then:

$$R^2 \equiv 12m_t^4 - 6m_W^4 - 3m_Z^4 \approx 11.5, \tag{5}$$

$$h \equiv 12m_t^2 - 6m_W^2 - 3m_Z^4 \approx 9.83, \tag{6}$$

$$L \equiv 12m_t^4 \ln m_t^2 - 6m_W^4 \ln m_W^2 - 3m_Z^4 \ln m_Z^2 \approx 0.743. \tag{7}$$

If we denote the masses squared of some hypothetic massive boson fields by $x_1, x_2, \ldots, x_n$ $(x_i > 0, \forall i)$ then their values should satisfy the conditions

$$\sum_{i=1}^{n} x_i^2 = R^2, \tag{8}$$

$$\sum_{i=1}^{n} x_i = h, \tag{9}$$

$$\sum_{i=1}^{n} x_i^2 \ln x_i \equiv \phi = L. \tag{10}$$

We shall now proceed as follows:

– Firstly we shall find a lower bound to the number of massive boson degrees of freedom due to the constraints (8), (9) which define a surface S in the space of the x_i.

– Secondly we shall study on S, for positive x_i, the extrema of the function ϕ by using the method of Lagrange multipliers. We shall derive analytic formulae for the extrema of the function ϕ on the constraint surface S and show that its minimum is much higher when the value of L coming from Eq. (7).

– Finally we shall obtain for ϕ, its minimum (and maximum) value as a function of the SM particle content plus possible additional fermions in order to investigate the conditions for the satisfaction of the condition in (10).

The constraints (8), (9) have a simple geometrical sense [4]: they describe a sphere and a plane in the n-dimensional space and their intersection S is an $(n-2)$-dimensional sphere, eventually to be sliced on the positivity boundary of the x_i. The distance of the plane from the origin of the coordinates is $h/\sqrt{n}$. In order to have an intersection between the sphere of radius R and the plane it is then necessary to have

$$n > h^2/R^2 \approx 8.4. \tag{11}$$

Thus, the number of massive bosonic degrees of freedom should at least be equal to 9. In general it is convenient to introduce the integer value n_0 for such a threshold

$$n_0 = \left\lfloor h^2/R^2 + 1 \right\rfloor, \tag{12}$$

so that $n \geq n_0$ is the requirement to have a non empty S. Now, in order to see when the condition (10) is also satisfied, it is convenient to calculate the minimum value of the function $\phi = \sum_{i=1}^{n} x_i^2 \ln x_i$ on the constraint surface S. Let us consider an auxiliary function

$$F(\{x_i\}) = \sum_{i=1}^{n} x_i^2 \ln x_i - \\ - \lambda \left(\sum_{i=1}^{n} x_i^2 - R^2 \right) - \mu \left(\sum_{i=1}^{n} x_i - h \right), \tag{13}$$

where λ and μ are the Lagrange multipliers. Searching for the extrema of the function F implies we should equate its derivatives with respect to x_i, λ and μ to zero. The last two conditions $\partial F/\partial\lambda$ and $\partial F/\partial\mu$ again give the constraints (8), (9). Differentiation with respect to x_i gives the system of equations:

$$x_i^2 \ln x_i - x_i - 2\lambda x_i - \mu = 0, \quad i = 1, \ldots, n. \tag{14}$$

Without loss of generality we can choose $x_1 \neq x_2$. Indeed it is possible to have $x_1 = \cdots = x_n$ if and only if $h^2/n^2 = R^2$, but this is a degenerate case, when the sphere and plane touch each other in only one point.

On substituting the values of x_1 and x_2 into the first two equations of the system (14), one obtains $\bar{\lambda}$ and $\bar{\mu}$ as functions of x_1 and x_2:

$$\bar{\lambda} = 1 + \frac{x_1 \ln x_1^2 - x_2 \ln x_2^2}{x_1 - x_2}, \quad \bar{\mu} = \frac{x_1 x_2 (\ln x_2^2 - \ln x_1^2)}{x_1 - x_2}. \tag{15}$$

Let us suppose that $\bar{x}_1, \cdots, \bar{x}_n, \bar{\lambda}, \bar{\mu}$ are a solution of the system (14) on S, i.e. with the constraints (8) and (9) already satisfied. On now substituting these values of $\bar{\lambda}$ and $\bar{\mu}$ into the $n-2$ remaining equations of the system (14) we can easily see that a solution is given by $x_1 = x_3 = x_4 = \cdots = x_{k+1}$ and $x_2 = x_n = x_{n-1} = \\ = \cdots = x_{k+2}$. This solution is a stationary point of the function F, or in other words the conditional stationary point of the function ϕ. Such a solution, with k coordinates having the value $\bar{x}_1 = x$ and the remaining $n-k$ coordinates the other value $\bar{x}_2 = y$, is given, as function of k and n, by

$$x = x(k,n) = \frac{h}{n} + \sqrt{\frac{R^2(n-k)}{nk} - \frac{h^2(n-k)}{n^2 k}}, \tag{16}$$

$$y = y(k,n) = \frac{h}{n} - \sqrt{\frac{R^2 k}{n(n-k)} - \frac{h^2 k}{n^2(n-k)}}, \tag{17}$$

where $1 \leq k \leq n-1$.

The values of x given by Eq. (16) are always positive, while the values of y can be negative. It is easy to show that the condition for the positivity of y is

$$k < n_0 \leq n. \tag{18}$$

We have seen that points of the type described above always satisfy the stationarity conditions (14) on the constraint surface. This does not mean that stationary points of other types cannot exist. Indeed, the analysis of the structure of Eq. (14) shows (see for details [9]) that, in principle, stationary points whose coordinates x_i have *three* different values can exist. However, if such points exist, at least one of these three values is negative and, hence, is of no interest to us. Thus, the minimum of the function ϕ can be reached *only* for the stationary points having the coordinates (16), (17) or on the boundary of the positivity region, where at least one of the coordinates x_i is equal to zero. For this last case the problem is reduced to one with lower dimensionality n.

If $n = n_0$ (the smallest possible value for the dimensionality of n) we notice that on the surface S all the x_i have positive values. Thus, the maximum and minimum values of the function ϕ on the constraint surface are obtained only for one of points with the coordinates x and y (see formulae (16), (17)).

Furthermore the following more general statement is true: for $n \geq n_0$ the maximum value of the function ϕ corresponds to $n = n_0$ and $k = 1$ while its minimum value corresponds to the point with $n = n_0$ and $k = n_0 - 1$. To prove it one may compute the derivatives of the function

$$\phi_1(k, n) = k\, x^2 \ln x + (n - k)\, y^2 \ln y, \qquad (19)$$

with respect to k and n. It can be shown [9] that $d\phi_1/dk < 0$ and $d\phi_1/dn > 0$ for the range of possibile physical values of k and n. In particular this means that the function $\phi_1(k, n)$ decreases with increasing k and has its minimum value at $k = n_0 - 1$ and $n = n_0$. This minimum value is

$$\phi_{1\ \min} = \bar{\phi}_1(R, h) = \\ = (n_0 - 1)x^2(n_0 - 1, n_0) \ln x(n_0 - 1, n_0) + \\ + y^2(n_0 - 1, n_0) \ln y(n_0 - 1, n_0), \qquad (20)$$

where one has to use Eqs. (12), (16) and (17). The solution of the equation

$$\sum_{i=1}^{n} x_i^2 \ln x_i = L \qquad (21)$$

exists on the constraint surface S only if

$$\bar{\phi}_1(R, h) < L. \qquad (22)$$

A direct calculation shows that for $n_0 = 9, \phi_{1\,\min} \approx 1.95$ which is higher than $L \approx 0.743$ in Eq. (7). For $n > n_0$ the minimum value of the function ϕ_1 is higher, so our first result is that it is not possible to have the cancellations in an extension of the SM obtained on only introducing new bosonic fields. One can also say that the value of the effective cosmological constant is proportional to $(\phi_{1\ \min} - L)$ and hence is of the order of 1 in terms of the top quark mass elevated to the fourth power. This is much higher when the observable value of the cosmological constant.

Let us now consider an extension which includes new fermionic fields with masses m_f in units of top mass and n_f degrees of freedom. On introducing the new parameters

$$\tilde{R}^2 = R^2 + \sum_f n_f\, m_f^4, \\ \tilde{h} = h + \sum_f n_f\, m_f^2, \qquad (23) \\ \tilde{L} = L + \sum_f n_f\, m_f^4\, \ln m_f,$$

one may search for solutions allowed by the inequality $\phi_1(k = n_0 - 1, n) < \tilde{L} < \phi_1(k = 1, n)$ for different $n \geq n_0$, by varying n_f and m_f. In general the lower bound for n slightly increases from n_0 depending on n_f and m_f. In the simplest case with only one additional fermion we already find solutions. Let us here give the results for three interesting cases and a non physical $(n_f \to \infty)$ case:

$$\text{Majorana}: \quad n_f = 2, \quad m_f > 1.52, \\ \text{Dirac}: \quad n_f = 4, \quad m_f > 1.46, \\ \text{Dirac quark (3 colors)}: \\ n_f = 12,\ m_f \in [0.3536, 0.3592] \cup \times \\ \times [0.4, 0.655] \cup [0.6892, 0.6914] \cup [1.4, \infty[, \\ n_f \to \infty, \quad m_f \in [\frac{1}{\sqrt{n_f}}, 0.853] \cup [1.355, \infty[\qquad (24)$$

More possibilities are allowed if one introduces more fermionic fields with different masses.

We finally investigated a simple extension of the SM obtained on introducing a Majorana fermion and assuming for the bosonic sector $n \ll N_B$, so that most of the bosons are practically massless and weakly coupled, as in the unparticle scenario mentioned near the end of the previous section. We found, as a function of the Majorana fermion mass, sets of solutions for the massive bosons (we consider the case $n = 10$). Of particular interest is the lightest boson mass which could play the role of the Higgs mass. In such a case we find for it the allowed mass intervals (in GeV): $[111, 139]$, $[115, 172]$, $[112, 178]$ and $[86, 177]$ for $m_f^2 = 2.5$, 3, 3.5 and 4 m_t^2 respectively. These are solutions compatible with the actual Higgs mass limits [6].

6. Conclusion. We have proved that it is impossible to construct a minimal extension of the SM by finding a set of boson fields which, besides cancelling the ultraviolet divergencies, can compensate the residual huge contribution of the known fermionic and bosonic fields of the Standard Model to the finite part of the vacuum energy density.

On the other hand we have found that the addition of at least one massive fermionic field is sufficient for the existence of a suitable set of boson fields which would permit the cancellations and we have obtained the allowed windows for the masses. This result is by itself very suggestive since in extensions of the SM often new extra fermions are considered, independently of any cancellation requirement. An example is the explanation by a see-saw mechanism of the smallness of the neutrino masses, which requires the presence of high mass Majorana neutrinos.

The addition of one Majorana or Dirac fermion requires a mass roughly at least 50% higher than the top quark mass. If the fermion belongs to a new quark fam-

ily, new mass windows appear (see (24)) with a range of values lower than the top quark mass, implying that this low mass family should be weakly coupled. Furthermore we have investigated numerically the simplest extensions of the SM which satisfy the constraints and found that the lightest massive boson can have a mass compatible with the bounds on the Higgs boson mass.

The problem we have addressed could also be studied in the context of renormalized effective actions wherein all parameters are running, including the cosmological constant as well as the field masses. Further the cosmological constant may also be affected by the presence of condensates which we do not consider in our analysis. Another point we feel is important, but not understood, is a finite vacuum energy contribution due to the presence of bound states, originating from interactions, whose conceptual distinction from fundamental particles may not appear obvious when one is trying to give an effective formulation at different scales. This is a typical feature of interacting quantum field theories wherein "fundamental" degrees of freedom appear to be different for different scales, one example being given by strong interaction physics. Again we believe that the most promising approach is given by a RG flow of effective actions. In this sense our results are only a first step in a more general scheme, which also takes into account any form of interaction.

However, we can already sketch an approach which could be developed to take into account the interaction between quantum fields. Firstly, let us notice that to treat the finite contribution of quantum fields to the vacuum energy it is reasonable to combine the cancellation mechanism proposed by W. Pauli [1] applied in [4] and in the present paper with the standard renormalization procedure for the ultraviolet divergences in the traditional quantum field theory of the scattering matrix (see e.g. [10]). Indeed, the cancellation of the contributions of boson and fermion fields to the vacuum energy is useful only to treat the one-loop divergences coming from propagators of free fields. As is well-known these divergences, which in standard quantum field theory, that is in the absence of gravity, are eliminated by the normal ordering procedure, create problems, when the gravity is included. All the other ultraviolet divergences need not be cancelled, but rather renormalized. It is worth remembering that even in the majority of models possessing an exact supersymmetry not all the ultraviolet divergences are cancelled. For example, in the Wess-Zumino model [11] the loop-contributions to the vertices are ultraviolet finite, while those to the propagators are divergent and should be renormalized.

Thus, the standard procedure of the elimination of ultraviolet divergences, results in the renormalization of masses, coupling constants and kinetic terms of the fields under consideration. The last (kinetic terms) can be fixed in such a way that they have standard form, i.e. the corresponding renormalization coefficients are equal to one [10]. The renormalization of the masses reduces to the appearance of the physically measurable masses – these are the ones which we have used for the sum rules in [4] and in the present paper.

The renormalization of the coupling constants (or the corresponding vertex diagrams) is more complicated. Here one should take into account the origin of the corresponding interactions. Indeed, the coupling constants coming from the self-interaction of non-Abelian Yang-Mills fields tend to zero in ultraviolet limit (the asymptotic freedom phenomenon [12]) and hence do not influence our sum rules for the cancellation of ultraviolet divergences. The electromagnetic interaction instead reveals the opposite phenomenon – the growth if the effective coupling constant in the ultraviolet region implying the existence of the so called Landau pole [13]. However, as a remedy to this problem one can use the analytic approach [14] (see also [15]), combining the renormalization group invariance with the Källen-Lehmann spectral representation. On applying this approach the Landau pole disappears and the coupling constant has a finite value in the ultraviolet limit and is well-defined. On now remembering that in spinor electrodynamics the only interaction term is represented by the cubic vertex multiplied by the coupling constant, one sees that its contribution to the vacuum energy is equal to zero and hence does not influence the rules for the cancellation of the ultraviolet divergences [4]. Thus, these sum rules formulated in [4] and used also in the present paper are valid in spite of the inclusion of interactions present in the Standard Model and its simple extensions. The problem of the contribution of interactions in the infrared region (finite contributions to the vacuum energy) where also bound states arise is more complicated and deserves further investigation.

To conclude, we wish to say that our results, using the present data, are encouraging since they suggest that reasonable non supersymmetric extensions of the SM with almost zero vacuum energy may exist.

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