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$\hbar$ as parameter of Minkowski metric in effective theory

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With the proper choice of the dimensionality of the metric components and matter field variables, the action for all fields becomes dimensionless. Such quantities as the vacuum speed of light c, the Planck constant $\hbar$, the electric charge e, the particle mass m, the Newton constant G never enter equations written in the covariant form, i.e., via the metric $g^{\mu\nu}$. The speed of light c and the Planck constant $\hbar$ are parameters of a particular two-parametric family of solutions of general relativity equations describing the flat isotropic Minkowski vacuum in effective theory emerging at low energy: $g^{\mu\nu}_{\rm Minkowski} = {\rm diag}(-\hbar^2, (\hbar c)^2, (\hbar c)^2, (\hbar c)^2)$. They parametrize the equilibrium quantum vacuum state. The physical quantities which enter the covariant equations are dimensionless quantities and quantities which have dimension of rest energy M or its power. Dimensionless quantities include the running coupling 'constants' α_i; the geometric θ-parameters which enter topological terms in action; and geometric charges coming from the group theory, such as angular momentum quantum number j, weak charge, electric charge q, hypercharge, baryonic and leptonic charges, number of atoms N, etc. Dimensionful parameters are mass matrices with dimension of M; gravitational coupling K with $[K]=[M]^2$; cosmological constant with dimension M^4; etc. In effective theory, the interval s has the dimension of $1/M$; it characterizes dynamics of particles in quantum vacuum rather than space-time geometry. The action is dimensionless reflecting equivalence between action and the phase of a wave function in quantum mechanics. We discuss the effective action, and the measured physical quantities including parameters of metrology triangle.

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1. Introduction. The system of units is based on the theoretical understanding of physical laws. The traditional approach to the system of units is based on the two great physical theories of the twentieth century: special relativity and quantum mechanics (QM), which suggest to fix the speed of light c to connect space and time units, and the Planck constant $\hbar$ to connect mass and time units [1]. In theoretical physics, these quantities are often considered as fundamental constants and are used as units: in these units, $c=\hbar=1$ [2]. However, another great theory – general relativity (GR) – undermines this approach.

Speed of light. In mechanics (Galilean, relativistic or any other) the energy of a freely moving body can be expanded in terms of small momentum:

$$E(\mathbf{p}) = M + K^{ik}p_ip_k + K^{ikmn}p_ip_kp_mp_n + ... \qquad (1)$$

In the Galilean invariant mechanics, (1) has only two terms, the internal or the rest energy M and kinetic energy with the isotropic mass tensor $K^{ik}=(2m)^{-1}\delta^{ik}$:

$$E(\mathbf{p}) = M + \frac{\mathbf{p}^2}{2m}. \qquad (2)$$

The Lorentz invariant special relativity connects two parameters of the body, inertial mass m and the rest energy M, via the speed of light,

$$m = \frac{M}{c^2}, \qquad (3)$$

and equation (1) is transformed to equation with the parameter of a body M and 'fundamental constant' c:

$$E^2 - c^2p^2 - M^2 = 0\,. \qquad (4)$$

GR removes 'fundamental constant' transforming (4) to

$$g^{\mu\nu}p_\mu p_\nu + M^2 = 0, \quad p_\mu = (-E, p_i). \qquad (5)$$

Equation (5) contains the parameter of a body, the rest energy M. The speed of light c becomes the part of the metric, and never enters explicitly any equation, which is written in the covariant form, i.e. when the equation is expressed in terms of metric field (see e.g. [3, 4]). It may enter only the solutions of equations, in particular as parameter of flat Minkowski space-time:

$$g^{\rm Minkowski}_{\mu\nu}(c) = {\rm diag}\left(-1, \frac{1}{c^2}, \frac{1}{c^2}, \frac{1}{c^2}\right). \qquad (6)$$

In fundamental theory the limiting speed is fundamental, which allows us to put $c=1$.

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In effective theory, the latter is problematic. For example, if the underlying microscopic system is anisotropic, the limiting speed of the low-energy excitations depends on the direction of propagation, and effective Minkowski space-time contains 3 parameters

$$g_{\mu\nu}^{\rm Minkowski}(c_1, c_2, c_3) = {\rm diag}\left(-1, \frac{1}{c_1^2}, \frac{1}{c_2^2}, \frac{1}{c_3^2}\right). \quad (7)$$

Such anisotropy of the physical speed of light will be revealed only at high energy. For the internal low-energy observers, the world obeys Lorentz invariance and equivalence principle of GR. The measured limiting speed is coordinate independent, isotropic and universal for all species (at least in the Fermi-point scenario of emergent gravity [3]), if observers do not use "xylophones, yachts and zebras to measure intervals along the x, y and z axes" [5], but use the same xylophone in all directions. Since xylophones (rods and clocks) are made of anisotropic quasipaticles, the rescaling of measured distances automatically occurs leading to apparent isotropy and universality. Isotropy emerges in the low-energy limit, while for external high-energy observer the speed of light depends on coordinates and energy and becomes different for different species.

Thus in both cases, fundamental and emergent, the limiting speed c drops out of any equation written in the covariant form. For fundamental GR this is trivial since one may put $c = 1$. In effective GR even such notion as the fundamental speed is simply absent and its introduction is artificial. Let us now show that the same occurs with another fundamental constant $\hbar$.

Planck constant $\hbar$. It relates the frequency of emitted photon with the energy levels of atom:

$$M_m - M_n = \hbar\omega_{mn}. \quad (8)$$

For an extended body the relation between the invariant mass M and the non-covariant frequency is coordinate dependent due to gravitational red shift. This means that $\hbar$ can be measured either in the ideal limit of a point object, which is a mathematical construction (string theory for example deals with extended objects) or in Minkowski space-time. This suggests that $\hbar$ is another characteristic of Minkowski vacuum, which can be absorbed by metric together with c.

This is achieved if the quantity $\hbar$, which traditionally is the prefactor in the quantum mechanical operator of momentum $p_\mu = -i\hbar\nabla_\mu$, is moved from p_μ to the metric $g_{\mu\nu}$ in Eq. (5). As a result, the isotropic Minkowski metric is characterized by two parameters, $\hbar$ and c, with $\hbar$ being the conformal factor of the Minkowski space-time:

$$g_{\mu\nu}^{\rm Minkowski}(c, \hbar) = \hbar^{-2}{\rm diag}\left(-1, c^{-2}, c^{-2}, c^{-2}\right). \quad (9)$$

Now when $\hbar$ is absorbed by the metric, it also does not enter any covariant equation. In particular, (8) becomes

$$M_m - M_n = \frac{\omega_{mn}}{\sqrt{g_{00}}}, \quad (10)$$

i.e. in GR the QM Eq. (8) is the version of the red shift Eq. (10).

With the choice of the dimensionality of the metric components in Eq. (9), the action becomes dimensionless, $[S] = 1$. An example is the dimensionless classical action for a freely moving massive particle:

$$S_M = M\int ds, \quad ds^2 = g_{\mu\nu}dx^\mu dx^\nu. \quad (11)$$

A dimensionless S leads to a natural formulation of QM in terms of Feynman path integral with the integrand e^{iS}. For a single particle, S is the phase of semiclassical wave function, while (5) becomes

$$g^{\mu\nu}k_\mu k_\nu + M^2 = 0, \quad k_\mu = (-\omega, k_i). \quad (12)$$

If gravity is fundamental and $\hbar$ is fundamental constant, the proposal to eliminate $\hbar$ from equations by hiding it into the metric (9) by scale transformation would be a formal mathematical trick only. Nothing prevents to use the scale factor larger or smaller than $\hbar$, if it is accompanied by the rescaling of mass M. If all the equations are covariant, one may safely set $c = \hbar = 1$.

If GR is effective and general covariance emerges only at low energy, then all 10 components of the metric become physical at higher energy, including the parameters c and $\hbar$, entering the Minkowski metric of the equilibrium vacuum. They are not universal and together with the other parameters of the metric may explicitly enter the higher order terms in the action, for which the covariance as emergent phenomenon is not valid. As a results the ratio $(M_m - M_n)/\omega_{mn}$, which is universal in the low-energy range, will depend on frequency at high energy. This makes impossible setting $\hbar = 1$.

In the q-theory, quantum vacuum is characterized by the "charge" q [6]. In the equilibrium self-sustained state of the vacuum – Minkowski vacuum – the charge q_0 satisfies equation $\epsilon(q) - q d\epsilon/dq = 0$, where $\epsilon(q)$ is vacuum energy. In principle, the q-theory may fix particular Minkowski state with parameters $c(q_0)$ and $\hbar(q_0)$.

Here we consider the lowest order terms in effective theory, and neglect the higher-order terms violating the emergent GR. We demonstrate that the choice (9) for Minkowski metric allows us to remove c and $\hbar$ from all the covariant equations without assumption that these quantities are universal. When c and $\hbar$ are inside the metric, the symmetry (gauge invariance, general covariance, etc.) does not prohibit the space-time dependence of c and $\hbar$. This is possibly the necessary step towards "quantum gravity", the microscopic theory in which c and $\hbar$ may and should depend on space-time. The extended version of this paper is in Ref. [7].

2. Effective action. Let us first discuss the effective action for the gauge fields and gravity as it appears, say, in the Fermi-point scenario [3], and rewrite it in such a way that $\hbar$ is hidden in the metric field. The main lesson is that in the low-energy corner, i.e. in the region of applicability of effective theory, the parameter $\hbar$ never enters explicitly.

Electromagnetic action. We choose the dimensions of the vector potential A_μ and field strength $F_{\mu\nu}$ as they naturally follow from the geometric origin of the gauge field. Since A_μ arises as a result of localization of the dimensionless $U(1)$ field, $\nabla_\mu \phi \to A_\mu$, one has:

$$[A_0] = [t]^{-1},\ [A_i] = [l]^{-1},\ [F_{ik}] = [l]^{-2},\ [F_{i0}] = [tl]^{-1}. \tag{13}$$

The interaction of a classical particle with electromagnetic field is

$$S_{\rm int} = q \int dx^\mu A_\mu. \tag{14}$$

Here q is the geometric charge of a particle corresponding to the $U(1)$ gauge group, with $q_e = -1$ for electron; $q_u = 2/3$ for up quark; $q_d = -1/3$ for down quark, etc. In QM, equation (14) corresponds to the covariant derivative $D_\mu = \nabla_\mu - iqA_\mu$. The motion equation of a massive particle with electric charge q in electromagnetic field follows from (11) and (14):

$$\frac{du^\mu}{ds} + \Gamma^\mu_{\lambda\sigma} u^\lambda u^\sigma = \frac{q}{M} F^{\mu\nu} u_\nu,\ \ u^\mu = \frac{dx^\mu}{ds}. \tag{15}$$

The action for the electromagnetic field is

$$S_{\rm em} = \int d^3x dt\ \frac{\sqrt{-g}}{16\pi\alpha}\ F_{\alpha\beta} F_{\mu\nu} g^{\alpha\mu} g^{\beta\nu}, \tag{16}$$

where the dimensionless parameter α is the logarithmically running coupling – the fine structure 'constant'. In effective theories, $1/\alpha$ naturally emerges as logarithmically divergent factor, and in principle it is space- and time-dependent. For example, in quantum electrodynamics with massless fermions which emerges in superfluid ^{3}He-A [3] one has $1/\alpha \propto \ln[1/(F^{\mu\nu}F_{\mu\nu})]$. With the choice (13), the actions (16) and (14) are dimensionless: $[S_{\rm int}] = [S_{\rm em}] = 1$; (14), (15) and (16) do not contain $\hbar$.

In effective theory, voltage (the difference of electric potentials) has the same dimension as frequency and electric current (the current of electrons): $[J_e] = [A_0] = [\omega] = [t]^{-1}$. The electric resistance and conductance are dimensionless: $[R] = 1$. This suggests the possibility that the dimensionless relations between voltage and frequency, between voltage and current, and between current and frequency may have quantized values in some systems. The corresponding Josephson effect, quantum Hall effect, and quantum pumping, which form the so-called metrology triangle are discussed in Sec. 4.

Action for gravity: The GR in effective theories is

$$S_{\rm grav} = \int d^3x dt \sqrt{-g} \left(\Lambda + \frac{K}{16\pi} \mathcal{R} + \ldots \right), \tag{17}$$

where K and Λ are gravitational coupling and cosmological constant respectively, and dots denote the higher order terms which include in particular the $\mathcal{R}^2$ terms. Using dimensions of the metric component

$$[g^{00}] = \frac{1}{[g_{00}]} = [\hbar]^2, [g^{ik}] = \frac{1}{[g_{ik}]} = [\hbar c]^2, [-g] = \frac{1}{[\hbar^8 c^6]}, \tag{18}$$

one obtains that the dimension of the curvature $\mathcal{R}$ is:

$$[\mathcal{R}] = [M]^2,\ \ [d^3x dt]\,[\sqrt{-g}] = [M]^{-4}. \tag{19}$$

As follows from effective theories, dimensions of the gravitational coupling and cosmological constant are

$$[K] = [M]^2,\ \ [\Lambda] = [M]^4, \tag{20}$$

while the $\mathcal{R}^2$ terms have dimensionless prefactors. As a result, the GR action is dimensionless, $[S_{\rm grav}] = 1$. The first two terms in (17) contain parameters with dimension M^n and thus they emerge only if conformal invariance is violated and the fundamental length or energy scale enters the underlying microscopic theory.

The dimension of the metric suggests that metric is not the quantity, which describes the space-time, but the quantity, which determines the dynamics of effective fields in the background of a given quantum vacuum.

Let us consider physical quantities which in principle can be measured, and express them in terms of the invariant parameters, which enter the action. They can be distributed in the following groups: (i) quantities, which contain $\hbar$ and c due to historical reasons, and do not contain these parameters when expressed in terms of the parameters emerging in effective theory; (ii) quantities, which still contain $\hbar$ even when expressed in terms of the effective theory parameters, but do not contain it after the rescaling of the metric in (18); (iii) dimensionless geometrical and topological charges.

3. Nontopological parameters. Effective theory emerging in the low energy corner contains such parameters as fine structure constant α, gravitational coupling

K, angular momentum quantum number j, charge quantum number q, and rest energies M of particles, etc. However, in the traditional description, which reflects the historical process of development of physical ideas, these quantities are splitted into the electric charge of a system Q, elementary electric charge e, speed of light c, Newton constant G, Planck constant $\hbar$, angular momentum J and masses m:

$$K = \frac{\hbar c^5}{G}, \quad \alpha = \frac{e^2}{\hbar c}, \quad M = mc^2, \quad j = \frac{J}{\hbar}, \quad q = \frac{Q}{e}. \tag{21}$$

In effective theories such splitting is not justified, and moreover it is not necessary since the measured quantities do not contain the traditional parameters explicitly. There are some examples below.

Electron energy in a Coulomb field. The energy levels of electron in the Coulomb field of proton [8]:

$$M_{n,j} = M_e \left(1 - \frac{\alpha^2}{2n^2} - \frac{\alpha^4}{2n^4}\left(\frac{n}{j+1/2} - \frac{3}{4}\right) + \ldots\right). \tag{22}$$

It is expressed via the rest energy of a free electron M_e; the fine structure constant α; quantum number n and angular momentum quantum number j.

Energy levels in a Newton potential. Rest energy of system of two point bodies interacting via Newton gravitational potential:

$$M_n = (M_1 + M_2)\left(1 - \frac{1}{2n^2K^2}\frac{M_1^3 M_2^3}{(M_1+M_2)^2} + \ldots\right). \tag{23}$$

It contains rest energies of bodies M_1 and M_2; and gravitational coupling K entering Einstein action in (17).

Black-hole temperature and entropy. Hawking temperature of a black hole with rest energy $M_{\rm BH}$ and its Bekenstein-Hawking entropy are:

$$T_{\rm BH} = \frac{K}{8\pi M_{\rm BH}}, \quad S_{\rm BH} = \frac{4\pi M_{\rm BH}^2}{K}. \tag{24}$$

Entropy of the rotating electrically charged black hole:

$$S(M,j,q) = \pi\left(2\frac{M^2}{K} - q^2\alpha + 2\sqrt{\frac{M^4}{K^2} - j(j+1) - q^2\alpha}\right). \tag{25}$$

Here j is the angular momentum quantum number of black hole; and q is its electric charge quantum number, the dimensionless charge determined by the gauge group, it is rational number with $q_e = -1$ for electron.

The above examples demonstrate that parameters Q, e, c, G, $\hbar$, angular momentum J and masses m are artificial. They reflect the long history of studies of the laws of physics, but do not appear in effective theories where all the physical laws naturally and simultaneously emerge in the low-energy corner.

Quantities which do not contain $\hbar$ after rescaling of metric, are considered on examples of Zeeman energy and Unruh effect.

Zeeman energy. The observable consequence of quantum electrodynamics is the Zeeman splitting of electron and muon energies in magnetic field B:

$$E_{\rm Zeeman} = \frac{B}{M}\left(1 + \frac{\alpha}{2\pi} + \ldots\right), \tag{26}$$

where M denotes either the electron M_e or muon M_μ rest energy. Here the field strength F_{ik} is given in geometric units in (13), $[F_{ik}] = [l]^{-2}$. The quantity $B^2 - E^2 = F_{\mu\nu}F^{\mu\nu}$ contains the metric elements, which enter $F^{\mu\nu}$. From dimensions of the metric elements in (18), the dimension of B is $[B] = [M]^2$, and (26) contains only the parameters M and α. There is one more parameter, the charge, which is not shown explicitly because for electron and muon one has $q_e^2 = q_\mu^2 = 1$. In traditional units one has (leaving only the first term): $E_{\rm Zeeman} = (e\hbar/m)B$.

Unruh effect. In effective theory, the Unruh temperature of the accelerated body is expressed via covariant acceleration with dimension $[a] = [M]$:

$$T_{\rm U} = \frac{a}{2\pi}\ , \ a^2 = g_{\mu\nu}\frac{d^2x^\mu}{ds^2}\frac{d^2x^\nu}{ds^2}, \tag{27}$$

$\hbar$ and c do not enter (27), being absorbed by $g_{\mu\nu}$. In traditional units $[a] = [l][t]^{-2}$, and $T_{\rm U} = \hbar a/2\pi c$.

4. Topological quantum numbers. Topological quantum numbers in action can be considered on the example of θ term, quantum Hall effect (QHE) and Josepson effect.

θ–term. θ term in action does not contain metric

$$S_\theta = \frac{\theta}{16\pi^2}\int d^3x dt\ e^{\alpha\beta\mu\nu}F_{\alpha\beta}F_{\mu\nu}. \tag{28}$$

It is automatically dimensionless, $[S_\theta] = 1$, due to equations (13) which reflects the geometric nature of gauge fields. The dimensional reduction of the θ-term gives the topological Chern-Simons term in the $2+1$ action:

$$S_{\rm CS} = \frac{k}{8\pi}\int d^2x dt e^{\alpha\beta\gamma}A_\alpha F_{\beta\gamma}, \tag{29}$$

where k is some fundamental number. $S_{\rm CS}$ is dimensionless and does not contain the metric field.

Quantum Hall effect. QHE in condensed matter is characterized by a similar $2+1$ action:

$$S_{\rm QHE} = \frac{q^2\nu}{8\pi}\int d^2x dt e^{\alpha\beta\gamma}A_\alpha F_{\beta\gamma}. \tag{30}$$

Here q is electric charge of fermion, which is $q = q_e = -1$ for electronic system. The dimensionless quantity ν is some fundamental number, integer of fractional, which characterizes the ground state of 2+1 fermionic system. In case of electrons, $q^2 = q_e^2 = 1$, electric current is

$$j^i = \frac{\delta S}{\delta A_i} = \frac{\nu}{2\pi} e^{0ij} F_{j0}. \qquad (31)$$

Being transverse to electric field it represents Hall current with quantized Hall conductivity:

$$\sigma^q = \frac{\nu}{2\pi}. \qquad (32)$$

In effective theory, the Hall and spin-Hall conductance are expressed via integer or rational number and π.

In traditional units, when A_0 is substituted by $e\tilde{A}_0/\hbar$ and S is multiplied by $\hbar$, the Hall conductivity [10]

$$\sigma_{xy} = 4\pi\alpha\sigma^q = 2\alpha\nu, \qquad (33)$$

where α is the fine-structure constant. In these units, σ_{xy} is not expressed via rational number and π only. That is why as distinct from σ^q, quantization of σ_{xy} is not exact, since α is not a constant but is a running coupling; it depends on the infrared cut-off and thus is space- and time-dependent. The failure of the traditional description to obtain exact quantization comes from the unjustified splitting of the vector potential, $A_\mu = (e/\hbar)\tilde{A}_\mu$, in the traditional description. As a result the field $\tilde{F}_{\mu\nu} = \nabla_\mu \tilde{A}_\nu - \nabla_\nu \tilde{A}_\mu$ is not gauge invariant: under gauge transformation $\tilde{A}_\mu \to \tilde{A}_\mu + (\hbar/e)\nabla_\mu \phi$, the field transforms as $\tilde{F}_{\mu\nu} \to \tilde{F}_{\mu\nu} + \nabla_\mu (\hbar/e) \nabla_\nu \phi - \nabla_\nu (\hbar/e) \nabla_\mu \phi$. The field $\tilde{F}_{\mu\nu}$ would be gauge invariant, only if e and $\hbar$ are fundamental constants. But the charge e is certainly not a fundamental constant. This is the so-called 'physical charge', which is obtained by splitting of α and thus is coordinate dependent. The gauge invariance and the true quantization of Hall conductivity are restored when the geometric vector potential A_μ and the geometric $U(1)$ charge of electron $q_e = -1$ are used.

Josephson effect in superconductors: The ac Josephson effect (JE) comes from the coupling of the phase ϕ of the order parameter with electromagnetic field: due to gauge invariance the time derivative of ϕ enters in action only in combination with the electric potential, $\partial_t \phi - qA_0$, where $q = 2q_e = -2$ is the electric charge of Cooper pairs. Taking into account the 2π periodicity of ϕ one obtains the Josephson relation:

$$\omega = 2\left(A_0^{(2)} - A_0^{(1)}\right), \qquad (34)$$

which contains only integer $|q| = 2$. This is because electromagntic field is given in geometric units, in which A_0 has dimension of frequency: $[A_0] = [\omega] = [t]^{-1}$. That is why the JE provides the standard of voltage in inverse unit of time. In traditional units, the Josephson relation contains parameters e and $\hbar$. The electric charge e is certainly non-fundamental, and the quantization is lost, as in the case of QHE.

Josephson effect in electrically neutral systems. In superfluid liquids, such as superfluid ^{4}He and ^{3}He, the role of voltage is played by gravitational field. In GR, for static gravitational field there is the Tolman law [9] for temperature and chemical potential. It relates local values $T(\mathbf{r})$, $\mu(\mathbf{r})$ with global values $T_{\rm T}$ and $\mu_{\rm T}$ which are space-independent in thermodynamic equilibrium:

$$T(\mathbf{r})\sqrt{-g_{00}(\mathbf{r})} = T_{\rm T}, \quad \mu(\mathbf{r})\sqrt{-g_{00}(\mathbf{r})} = \mu_{\rm T}, \qquad (35)$$

The dimensions of the local and global quantities are

$$[T] = [\mu] = [M], \quad [T_{\rm T}] = [\mu_{\rm T}] = [\omega] = [t]^{-1}. \qquad (36)$$

Josephson oscillations emerge due to difference of chemical potentials, but they cannot emerge in equilibrium: they only occur if the Tolman potential have a jump across the Josephson junction. That is why the correct Josephson relation in neutral superfluids should be:

$$\omega = \nu\left(\mu_{\rm T}^{(2)} - \mu_{\rm T}^{(1)}\right), \qquad (37)$$

where integer $\nu = 1$ for superfluid ^{4}He and $\nu = 2$ for superfluid ^{3}He with pairing mechanism of superfluidity. In traditional units, the parameter $\hbar$ enters explicitly, and if $\hbar$ is not fundamental, the quantization is not exact. This in principle allows us to experimentally resolve between fundamental and non-fundamental $\hbar$.

In terms of the local chemical potential μ, equation (37) becomes (we use $\nu = 1$ assuming superfluid ^{4}He):

$$\omega = \sqrt{-g_{00}^{(2)}}\,\mu^{(2)} - \sqrt{-g_{00}^{(1)}}\,\mu^{(1)}. \qquad (38)$$

In case when gravity is the same across the junction, and only the values of the local chemical potential μ are different on two sides of the contact, (38) becomes

$$\omega = \sqrt{-g_{00}}(\mu^{(2)} - \mu^{(1)}). \qquad (39)$$

In traditional units (39) is $\omega = (\mu^{(2)} - \mu^{(1)})/\hbar$.

In the other case, when the chemical potential is the same on both sides of the contact, but the gravitational potential is different, the equation (38) becomes

$$\omega = \mu\left(\sqrt{-g_{00}^{(2)}} - \sqrt{-g_{00}^{(1)}}\right). \qquad (40)$$

This situation takes place when the Josephson junction is represented by a channel connecting two vessels in which the level of the liquid has different height, h_2 and h_1. For a weak gravitational potential in the background of Minkowski space-time, one has $g_{00} \approx -(1 - 2\Phi/c^2)/\hbar^2$, $\sqrt{-g_{00}^{(2)}} - \sqrt{-g_{00}^{(1)}} \approx -(\Phi^{(2)} - \Phi^{(1)})/\hbar c^2$, $\Phi^{(2)} - \Phi^{(1)} = g(h_2 - h_1)$, and (40) transforms to the familiar expression for Josepson effect in gravitational field:

$$\hbar\omega = mg|h_2 - h_1|. \quad (41)$$

In superfluid ^{4}He at $T = 0$, the chemical potential equals the rest energy per unit ^{4}He atom: $\mu = M$. It slightly differs from the rest energy of an isolated ^{4}He atom, $M \neq M_4$, due to the energy added by interaction between the atoms and zero point motion. That is why in effective theory, the Josephson relation (39) has the same form as both the gravitational red shift and the energy-frequency relation in QM (10):

$$\omega(\mathbf{r}) = \sqrt{-g_{00}(\mathbf{r})}\left(M^{\rm i} - M^{\rm f}\right), \quad (42)$$

where $M^{\rm i}$ and $M^{\rm f}$ are the rest energies of an atom in initial state before radiation of photon and in the final state after the radiation correspondingly.

Metrology triangle. Another topological effect is quantum pumping (QP), the transfer of fermions by periodic change of the parameters of the system: $\dot{N} = \nu\omega/2\pi$, where $\dot{N}$ is the number of fermions transferred per unit time between two subsystems, and ν is topological quantum number. The QP in electronic systems (single-electron tunnelling [10, 11]) reflects the quantization of the number of electrons. Since electric current is the charge q transferred per unit time, $J = q_e\dot{N}$, one has relation between current and frequency: $J = q_e\nu\omega/2\pi$. QP allows to calibrate frequency by measuring the current J, or to calibrate current by tuning frequency. QP completes the so-called metrology triangle: JE, QHE and QP relate repsectively voltage and frequency, current and voltage, and frequency and current.

5. Schrödinger equation. *Klein-Gordon equation for scalar field.* In effective theory, the dimensionless action for a scalar field $\Phi(x)$ with dimension $[\Phi] = [M]$ has the form:

$$S_{\rm scalar} = \frac{1}{2}\int d^3x dt\, \sqrt{-g}\, \left(g^{\mu\nu}\nabla_\mu\Phi^*\, \nabla_\nu\Phi - M^2|\Phi|^2\right). \quad (43)$$

The kinetic term has the same dimension as the mass term without artificial introduction of the parameter $\hbar$.

Schrödinger equation. The nonrelativistic Schrödinger action can be obtained by expansion of Eq. (43). In case of space-time independent g^{00} one introduces the Schrödinger wave function Ψ with dimension $[\Psi] = [M]^{3/2}$

$$\Phi(\mathbf{r}, t) = \frac{1}{\sqrt{M}}\exp\left(iMt/\sqrt{-g^{00}}\right)\Psi(\mathbf{r}, t). \quad (44)$$

After expansion over $1/M$ one obtains

$$S = \int d^3x dt\sqrt{-g}L,$$
$$2L = i\sqrt{-g^{00}}\left(\Psi\partial_t\Psi^* - \Psi^*\partial_t\Psi\right) + + \frac{ig^{0k}}{\sqrt{-g^{00}}}\left(\Psi\nabla_k\Psi^* - \Psi^*\nabla_k\Psi\right) + \frac{g^{ik}}{M}\nabla_i\Psi^*\nabla_k\Psi. \quad (45)$$

For Minkowski space-time, (45) transforms after renormalization $\Psi \to \Psi(\hbar c)^{3/2}$ to the Schrödinger action

$$S = \frac{1}{2}\int d^3x dt\left(i\left(\Psi\partial_t\Psi^* - \Psi^*\partial_t\Psi\right) + \frac{\hbar}{m}\nabla_i\Psi^*\nabla_i\Psi\right). \quad (46)$$

The Eq. (46) contains a single parameter: the ratio of parameters $\hbar$ and m. In non-relativistic QM, they always enter together. Examples are Eq. (41) for the Josephson effect in neutral superfluids and also the quantum of circulation of superfluid velocity: $\kappa = 2\pi\nu\hbar/m$. Here m is the mass of an atom in a superfluid liquid; $\nu = 1$ for superfluid ^{4}He; $\nu = 1/2$ for superfluid ^{3}He-B; and $\nu = 1/4$ for the four-particle correlated state, see e.g. Sec.10 in Ref. [12].

The traditional expression for non-relativistic Schrödinger action is obtained after multiplication of the action (46) by $\hbar$. It contains the parameters of the equilibrium Minkowski vacuum, c (via $m = M/c^2$) and $\hbar$. In the traditional form the spectrum of a nonrelativistic particle with mass m in, say, 1D box of size L_x with inpenetrable walls is

$$E_n = \frac{\hbar^2\pi^2n^2}{2mL_x^2}. \quad (47)$$

However, being written in the covariant form

$$E_n = \frac{\pi^2n^2}{2M(\Delta l)^2}, \quad (48)$$

where Δl is the proper distance, $(\Delta l)^2 = L_x^2 g_{xx}$, the energy spectrum does not contain parameters m, c and $\hbar$ explicitly. This demonstrates that even the nonrelativistic QM contains the rest energy M instead of traditional mass m. This suggests that the Lorentz invariance and general covariance are not very important factors. The metric field $g_{\mu\nu}$ may serve as the source of parameters $\hbar$ and c even for the 'one half' of GR [13], when the matter fields feel the metric field as effective geometry, but $g_{\mu\nu}$ does not necessarily obey GR.

6. Discussion. Emergent/fundamental. In GR, parameters $\hbar$ and c vanish from covariant equations. This statement is trivial or not, depending on whether GR is fundamental or emergent.

In fundamental physics, $\hbar$ is a universal constant, that is why, in principle it can be hidden in the metric. After it is included into the metric, $\hbar$ completely disappears from all physical equations if the action is represented in terms of the natural parameters, and becomes the second parameter of a particular two-parametric family of solutions of GR equations describing the flat isotropic Minkowski vacuum in (9). After $\hbar$ becomes the part of the metric, it shares the same fate as parameter c. Since GR does not discriminate between Minkowski metrics, $\hbar$ together with c becomes the matter of convention. If GR is fundamental, the inclusion of $\hbar$ into the metric is equivalent to the choice of units $\hbar = c = 1$, which does not lead to new results.

In the emergent theories, there are no universal constants: the 'fundamental' constants may be given by vacuum expectation values of the q-field [6], of scalar fields [5], of a 4-index field strength [5, 6], etc. That is why $\hbar$ and c could and should be space/time dependent in the same manner as α. At first glance, the space-time dependence of $\hbar$ would violate many physical laws. This happens if the equations are written in the traditional from. The traditional form of the Feynman integral over the fields χ is $\int d\chi \exp{(iS/\hbar)}$. The quantum-mechanical phase factor is a compact U(1) quantity, which means that action acquires topological properties. If $\hbar$ is space-time dependent, it must be introduced within the integral: $S/\hbar \rightarrow \int d^3x dt\ \hbar^{-1}\sqrt{-g}L$. But this would violate general covariance (if $\hbar$ is not a scalar field). That is why in emergent GR, it is simply necessary to include $\hbar$ into the metric in order to avoid such violations. As a result the action becomes dimensionless, and the Feynman integral becomes $\int d\chi \exp{(iS)}$. Just in the same way the gauge invariance requires the use of the geometric electric charge q instead of the traditional electric charge e, which was introduced at the earlier stage of physics. The latter disappears from equations if geometric vector potential A_μ and fine structure 'constant' α are used.

In emergent GR, $\hbar$ and c do not enter equations of GR, but they enter the special solution of the equations, which describes the effective Minkowski metric emerging in the background of the equilibrium state to which the vacuum relaxes in our part of the Universe.

In conclusion, the fundamental GR is equivalent to the universal $\hbar$, while in emergent GR $\hbar$ is non-universal. In the former case $\hbar$ can be included into metric, while in the latter case $\hbar$ must be included into metric. Inclusion of $\hbar$ into metric suggests that $\hbar$ becomes related to gravity (maybe to dilaton field in effective GR) and this reflects the peaceful coexistence of GR and QM.

Finally, let us mention such an effective GR, where it is impossible to put $\hbar = 1$. If Minkowski vacuum emerges as a result of spontaneous symmetry breaking, the symmetry breaking leads to domain walls separating the degenerate vacua with different signs of $\hbar$ and c. For fermions such vacua are physically different, because fermionic dynamics is determined by a vierbein which depends linearly on $\hbar$ and c. Within the $\hbar$-wall, the parameter $\hbar$ crosses either the value $\hbar = 0$ or the value $\hbar = \infty$, both forbidding using the convention $\hbar = 1$. In condensed matter the analogs of such vierbein walls are the walls in which one, two or all the three speeds of light in Eq. (7) change sign [14].

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