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Weak solution for the Hele-Shaw problem: viscous shocks and singularities

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In Hele-Shaw flows a boundary of a viscous fluid develops unstable fingering patterns. At vanishing surface tension, fingers evolve to cusp-like singularities preventing a smooth flow. We show that the Hele-Shaw problem admits a *weak solution* where a singularity triggers *viscous shocks*. Shocks form a growing, branching tree of a line distribution of vorticity where pressure has a finite discontinuity. A condition that the flow remains curl-free at a macroscale uniquely determines the shock graph structure. We present a self-similar solution describing shocks emerging from a generic (2,3)-cusp singularity – an elementary branching event of a branching shock graph.

1. Introduction. Hele-Shaw flow describes a 2D viscous incompressible fluid with a free boundary at low Reynolds numbers. The fluid is either sucked out from a drain or driven to a drain by another, inviscid, incompressible liquid [1]. Darcy law governs the viscous flow:

$$\mathbf{j} = -K\nabla p, \quad \mathbf{j} = \rho_0 \mathbf{v}. \tag{1}$$

If density ρ_0 and hydraulic conductivity K are constants, then pressure in the incompressible fluid p is a harmonic function, $\Delta p = 0$. At a drain (set at infinity), the fluid disappears with a constant flux $Q = \oint_\infty \mathbf{j} \times \mathrm{d}\ell$.

At vanishing surface tension (the case we consider), pressure is a constant along a boundary. Thus, in the fluid, pressure is a solution of the Dirichlet boundary problem. The boundary itself evolves in time.

A compact form of the law involves only a boundary: normal velocity of a line element of the boundary $\mathrm{d}\ell$ is proportional to its harmonic measure $\mathrm{v}\mathrm{d}\ell \sim \mathrm{d}\mu$ (Harmonic measure is a distribution of a Brownian excursion (emanating from a drain) as it hits the boundary $\mathrm{d}\mu = |\nabla p|\,\mathrm{d}\ell$).

In this form, the Darcy law goes far beyond applications to fluid dynamics [2]. It is closely related to a wide class of 2D growth and solidification processes such as DLA [3], flows in granular media [4], visco-elastic flow [5], evaporative patterns of fluid monolayers [6], etc.

Common patterns observed in these flows are characterized by intricate viscous fingering instabilities [7].

As such the problem is ill-defined: at a finite, *critical time* fingers develop cusp singularities (points of infinte curvature) [8, 9, 10], when the Darcy law stops making sense. Nevertheless, in many important physical problems such as flows in granular media and solidification, which at large scales are described by Darcy law, flows are not limited and go over a singularity. Resolving finite-time singularities in a physical manner, and a description of a flow beyond singularities is a major long-standing problem in the field.

An origin of singularities is the approximation of all physical parameters which could stabilize a flow at a microscale being set to zero. While this is a valid approximation at a smooth flow, at a singularity perturbations are singular, and an order of limits when different physical parameters are brought to zero is essential. One perturbation regularizing a flow is surface tension. However, interesting patterns are observed in processes where surface tension is either small, or does not exist at all, like in solidification and granular materials. This is the regime we are interested in. We assume that the surface tension (if any) must be set to zero first, before other parameters such as compressibility, are set to zero as well.

In this Letter we present what we believe might be the solution to the problem of finite-time singularities in the Hele- Shaw problem.

We impose the incompressibility and curl-free conditions at a large scale, but relax them at a microscale, setting surface tension to zero in the first place. Under this setting the Hele-Shaw problem admits a *weak solution*: a singular cusp-like finger triggers *shocks* – lines of discontinuity for pressure.

Shocks propagate through the fluid, forming a growing, *branching tree*. In this Letter we analyze a generic cusp-singularity giving rise of an elementary branching event of what will become a complicated degree-two tree of shocks.

The solution describing local origin of viscous shocks or any further branching event of already existing shocks pattern is self-similar and does not depend on the details of the flow.

A distinct feature of the Hele-Shaw flow is integrability [11 – 18]. Our weak solution is the only regularization of singularities which preserves the flow integrability.

A comment is in order: the most common appearance of shocks in hydrodynamics are ultrasonic flows in compressible fluids at vanishing viscosity. Formally they are caused by inertial terms in hydrodynamic equations. Shocks in Hele-Shaw flow considered here are of a different nature. They occur at vanishing Reynolds numbers when inertial terms are negligible and viscosity is high. We refer to them as *viscous shocks.*

2. Analytic and integrated forms of the Darcy Law We recall a description of a viscous finger in terms of a ***height*** function [16]. In Cartesian coordinates aligned with a finger axis (Fig.1), a finger is given by a graph

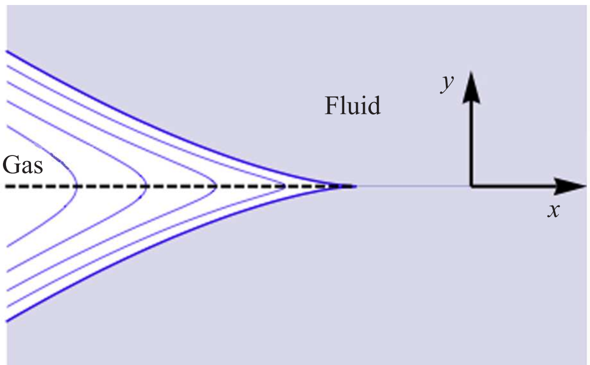


Fig.1. A growing finger in local Cartesian coordinates. The dashed line is the branch cut of the height function

$y(x)$. Let $X = x + iy$ be a complex coordinate of the fluid. The height function $Y(X, t)$ defined in the fluid is an analytic function whose boundary value on the fluid boundary is $Y(X)_{X=x} = y(x)$. Its discontinuity across the branch cut is $\mathrm{disc}\,\overline{Y}|_{X=x} = 2y(x)$. In terms of the height function the Darcy law reads:

$$\rho_0 \dot{Y} = -\partial_X \phi, \tag{2}$$

where the analytic function $\phi = \psi + \mathrm{i}p$ is a complex potential of the flow, and ψ is a stream function.

An important physical characteristic is the (time-depedent) capacity of the flow. It is defined as $C(t) = \frac{1}{2\pi\mathrm{i}} \oint_\infty \phi\, \mathrm{d}\phi$, where the integral goes around a drain. According to Darcy law, the power required to drain the fluid $N(t) = \frac{1}{2\pi K} \oint_\infty p\,(\mathbf{j} \times \mathrm{d}\ell)$ yields capacity $N(t) = -C$.

Integrated form of Darcy law: The integral $\Omega(X) = -\mathrm{i}\int_e^X \rho_0 Y \mathrm{d}X$ gives yet another form of the Darcy law:

$$\dot{\Omega} = \mathrm{i}\phi = -p + \mathrm{i}\psi. \tag{3}$$

Let us integrate (2) over a cycle B in the fluid:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Im} \oint_B \mathrm{d}\Omega = \oint_B \mathbf{j} \times \mathbf{d}\ell = \oint_B \mathrm{d}\psi, \tag{4}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Re} \oint_B \mathrm{d}\Omega = -\oint_B \mathbf{j} \cdot \mathbf{d}\ell = -\oint_B \mathrm{d}p = 0. \tag{5}$$

The imaginary part measures a flux of fluid through the cycle, the real part measures circulation along the cycle. The latter vanishes. At infinity, $\mathrm{i}\oint_\infty \mathrm{d}\Omega = Qt$ represents the mass of fluid drained up to time t.

3. Singularities of the Hele-Shaw flow. The pressure gradient is highest where the boundary curvature is large. Thus by (1), growth velocity is largest (and increasing) at finger tips. It diverges at a critical time; the finger then becomes a cusp of type $(2, 2l + 1)$: $y^2 \sim x^{2l+1}$ [8, 9, 19, 10, 16], Fig.1.

Cusps of types $(2, 4k - 1)$ and $(2, 4k + 1)$ evolve differently. In [17] it was shown that for the type $(2, 4k+1)$, a new droplet of inviscid fluid emerges next to the finger tip, before it evolves into a cusp. The fluid becomes multiply-connected, but evolution continues smoothly.

No continuous evolution is possible for $(2, 4k{-}1)$ cusps, including the most generic cusp (2,3) – subject of this paper.

4. Hydrodynamics of a critical finger. A finger approaching the (2,3)-cusp is especially simple [15]. Fixing a scale and the origin, it is given by:

$$-Y^2 = 4X^3 - g_2 X - g_3 = 4(X - e_3)(X - e_2)(X - e_1), \tag{6}$$

where g_2 and g_3 are real time dependent coefficients. One of the branch points, e_3, may be chosen real. It is the tip of a finger. The other two are conjugated $e_1 = \bar{e}_2$. The coefficient $g_2 = 4(|e_1|^2 - e_3^2)$ is determined by the drain rate, Q. We set the rate that $g_2 = -12t$, and count time from a cusp event: flow is smooth at $t < 0$.

In this normalization, capacity equals $C = -\dot{g}_2$. Well before a critical time, $e_3 < 0$. Then at $t < 0$, $g_2 > 0$ and $\mathrm{Re}\, e_1 > 0$: the two branch points are located in the fluid. This breaks the conditions of incompressibility, unless the roots coincide to a real double point: $e_1 = e_2$. Thus, $g_3 = 8(-t)^{3/2}$. The height function at $t < 0$ is a degenerate elliptic curve [15]:

$$Y^2 = -4\,(X - e(t))\,(X + e(t)/2)^2\,, \quad e(t) = -2\sqrt{-t}. \tag{7}$$

A finger becomes a cusp when the branch point e and double point $-e/2$ merge to a triple point. An important property of the critical finger (7) is that it is self-similar:

$$Y(X,t) = |t|^{3/4} Y\left(|t|^{-1/2}X, 1\right). \tag{8}$$

5. Weak form of the Hele-Shaw flow. Once the flow reaches a cusp singularity, it is no longer governed by the differential form of the Darcy law. This situation is typical for conservation laws of hyperbolic type $\partial_t u + \partial_z f(u) = 0$. There as well, smooth initial data develop into a shock at a finite time. Shocks occur to ill-defined conservation equations which arise as approximations of a well-defined problem. Adding a deformation through terms with higher gradients prevents formation of singularities. However, a smooth solution of a deformed equation may become discontinuous as a deformation is removed. It is then called a *weak solution* [20]. Validity of the differential equation on both sides of a shock determines a traveling velocity of a front $V = \frac{\text{disc}\, f}{\text{disc}\, u}$ - the Rankine-Hugoniot condition [21].

Often, physical principles determine dynamics of shocks without specific knowledge of the deformation used, such that different deformations lead to the same weak solution. The best known example is the Maxwell rule determining the position of Burgers-type shock fronts [21].

Darcy law is a conservation law of hyperbolic type, and we adopt the same strategy. We will be looking for a weak solution of the Hele-Shaw problem when the Darcy law is applied everywhere in the fluid except on a moving, growing and branching graph $\Gamma(t)$ of viscous shocks (or cracks), where pressure suffers a finite discontinuity.

A few natural physical principles guide toward a unique weak solution. We give three equivalent formulations.

The first formulation is in terms of the height function, treated as a complex vector:

- The canonically oriented (anti-clockwise) discontinuity of the height function's complex conjugate is parallel to the shock line directed such that:

  $$\rho_0\, \text{disc}\, \overline{Y}|_\Gamma = -2\sigma\boldsymbol{\ell}, \quad \sigma = \rho_0|Y| > 0\ , \tag{9}$$

  where $\boldsymbol{\ell}$ is a unit vector canonically oriented along the shock line.

An equivalent invariant formulation is in terms of the generating function (3):

- Discontinuity of Ω on shocks is imaginary; $\text{Re}\, \text{disc}\, \Omega$ is *increasing* away from both sides of a shock,

  $$\text{Re}\, \text{disc}\, \Omega|_\Gamma = 0, \quad \text{Re}\, \Omega|_{X\to\Gamma} > 0. \tag{10}$$

The first condition in (10) is equivalently written as $\text{Re}\, \mathrm{i} \oint Y \mathrm{d}X = 0$ for all cycles. Curves of this kind are called Krichever-Boutroux curves. They appeared in studies of Whitham averaging of non-linear waves [22] and asymptotes of orthogonal polynomials [23]. Neither appearance is coincidental.

The third formulation is in hydrodynamics terms.

6. Hydrodynamics of viscous shocks. We require that the fluid is irrotational at macroscale, but allow vorticity at a microscale – i.e. at a scale set by a vanishing regularizing parameter.

The first conditions of (9), (10) insure that the fluid remains *curl-free* despite of shocks. Let us chose a contour B including a portion of a shock. Then (9), (10) mean that the integral (5) still vanishes despite of discontinuities of Ω and pressure.

Let us study this condition in detail. The time derivative has two contributions: one from evolution of the height function $Y(X,t)$ at a fixed X, another from motion of shocks. Denote the shock front velocity by $\mathbf{V}_\perp$ (normal to the front, directed along the vector $\mathbf{n} = -\mathrm{i}\boldsymbol{\ell}$). Then the time derivative in (5) reads $\text{Im}\,[\text{disc}\, \dot{Y}\, \boldsymbol{\ell} + \nabla_\parallel(\text{disc}\, Y \cdot V_\perp)]_\Gamma = 0$, where $\nabla_\parallel$ represents the derivative along the direction tangent to the front along the vector $\boldsymbol{\ell}$. The first term is a jump of the velocity of the fluid parallel to a shock $-\text{disc}\, v_\parallel$. The second term is purely real. It equals $\nabla_\parallel(\sigma\mathbf{V}_\perp)$. Together, they yield to the condition:

$$\nabla_\parallel J_\perp + \text{disc}\, j_\parallel = 0, \quad J_\perp = \sigma\mathbf{V}_\perp, \quad j_\parallel = \rho_0\, v_\parallel. \tag{11}$$

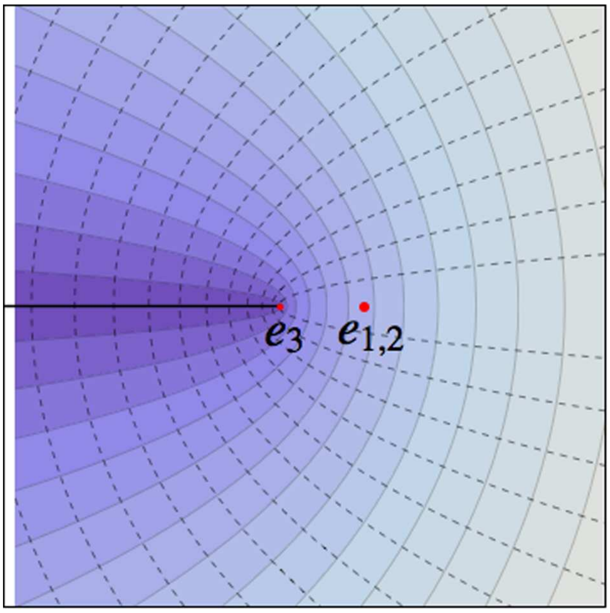
The first term in this equation represents the transport of mass by a shock (normal to the shock), while the second is a circulation of the surrounding fluid flow. They compensate each other.

This condition, derived solely from the requirement that σ is real, suggests to interpret viscous shocks as a single layer of microscale vortices with a line density σ and fixed orientation. Therefore, the weak solution describes a fluid with zero vorticity at a macroscale: vorticity concentrated in shocks is compensated by the circulation of fluid around shocks.

Using Darcy law we replace the fluid velocity in (11) by $-\nabla_\parallel p$, and integrate (11) along the cut. We obtain the Rankine- Hugoniot condition:

$$\sigma\mathbf{V}_\perp = (\text{disc}\, p)\, \mathbf{n}\ . \tag{12}$$

The second set of conditions in (9), (10) ($\sigma > 0$) imply that shocks move toward higher pressure, i.e, represent a deficit of fluid - cracks. This follows from the curl-free condition and is consistent with an assumption that at infinity (at a remote part of a finger, and at a drain), the flow is not affected by shocks.



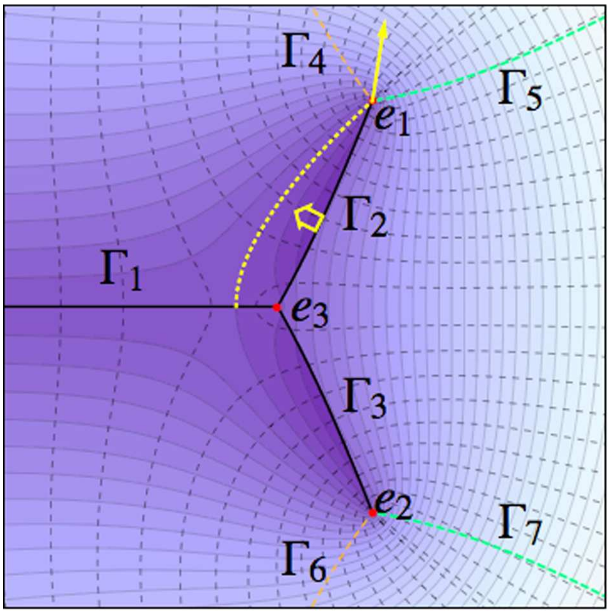


Fig.2. The equi-pressure lines and flow lines (dashed) before the transition (left), and after the transition (right). Pressure gets larger as the shade gets darker. Before the transition $e_2 = e_1$ is the double point. The thick lines (on the right panel) connecting the branch points are the shocks. The orange and green dashed lines are not admissible the level lines of $\mathrm{Re}\,\Omega$. The fluid flows to the lighter region toward low pressure. Shocks move toward the darker region (higher pressure). A bright dotted line is the zero-pressure line. The arrows are moving directions of the shock and the branch point. On the left of the zero-pressure line the finger expands, on the right the finger retreats

Consider the integral around the drain i $\oint_\infty \mathrm{d}\Omega = Qt$. Before the transition, a contour of integration can be smoothly deformed to the fluid boundary. It yields the total mass $2\rho_0 \oint y(x)\mathrm{d}x = \rho_0 \iint \mathrm{d}y\mathrm{d}x$. After the transition, the integral acquires a contribution from shocks: $-\oint_\Gamma \sigma \mathrm{d}\ell$. Viscous shocks can be seen as a single layer mass deficit: density of the fluid is no longer constant, $\rho(X) = \rho_0 - \delta_\Gamma(X)\sigma(X)$ (here δ_Γ is the delta-function on shocks).

The Rankine-Hugoniot condition (12), curl-free condition (9) and the differential form of the Darcy equation (2) combined give the *weak form of the Darcy law*. Remarkably, these conditions uniquely determine a shock pattern. Shock graphs with one and two branching generations are represented in Figs.2, 3. We will show how they work by an analysis of the (2,3)-cusp.

7. Self-similar weak solution: beyond the (2,3)-cusp. In the remaining part of the Letter we briefly describe a solution to the most generic singularity, (2,3).

Before becoming cusps, fingers are described by (7). The height function is a self- similar degenerate elliptic curve. Two branch points located in the fluid coincide to a double point. At a critical time, the tip meets the double point. After the critical time, we look for a weak solution, allowing pressure to have finite discontinuities on some curves. We assume that infinity is not affected by the transition, such that the height function is still an elliptic curve (6) with $g_2 = -12t$, but now $g_2 < 0$. We have only to find g_3 from the condition (10). Solution is again self- similar, $g_3(t) = g_3(1)t^{3/2}$. Once $g_3(1) \neq 8$, the curve is not degenerate anymore. The double point splits into two branch points, $e_1 \neq e_2$. They appear in the fluid as endpoints of shocks.

Since branch points are simple, all quantities have opposite signes on opposite sides of shocks: $\mathrm{disc}\,\Omega = 2\Omega|_\Gamma$. Then condition (10) means that $\mathrm{Re}\,\Omega$ vanishes on shocks. Since the endpoints belong to the fluid, by virtue of (3), pressure also vanishes. The branch point e_3 is a tip of a finger where pressure obviously vanishes. This is the governing condition:

$$p(e_{1,2}) = \mathrm{Im}\,\phi(e_{1,2}) = 0. \qquad (13)$$

The scaling property (8) is sufficient to express the potential ϕ and Ω through elliptic integrals:

$$\phi(X) = 6\int_{e_3}^{X}\left(X + \frac{3}{2}\frac{g_3}{g_2}\right)\frac{dX}{Y(X)}, \qquad (14)$$

$$\Omega(X) = -\mathrm{i}\int_{e_3}^{X} Y\,dX = -\frac{2\mathrm{i}}{5}(XY - 2t\phi). \qquad (15)$$

The governing equation (13) is then expressed through complete elliptic integrals K and E [24]:

$$(16m^2 - 16m + 1)E(m) = (8m^2 - 9m + 1)K(m) \ , \quad (16)$$

where

$$m = \frac{1}{2} + \frac{3}{2}\frac{1}{\sqrt{9+4h^2}}, \quad \frac{3}{2}\frac{g_3}{g_2 t^{1/2}} = 3\sqrt{\frac{3}{4h^2-3}\frac{4h^2+1}{4h^2-3}}.$$

Solution of (16) is transcendental [24]:

$$-\frac{e_{1,2}}{e_3} = \frac{1}{2} \pm ih, \quad h = 3.24638225374\ldots \quad (17)$$

It determines the moving ends of shocks.

8. Discontinuous change of capacity. The genus transition is summarized by an abrupt change of $g_3|_{t=\pm 1}$. We express this fact in normalization-independent terms using time derivative of the capacity. The ratio

$$\eta = \lim_{t\to 0}\frac{\dot{C}_{t>0}}{\dot{C}_{t<0}} = \frac{3}{2}\frac{g_3}{g_2}t^{-1/2} = 0.91522030388\ldots \quad (18)$$

is a unique *universal* number describing the transition.

9. Flow and shocks. Shocks are anti-Stokes lines of Ω, i.e., zero-level lines of $\mathrm{Re}\,\Omega$ selected by the admissibility condition (9, 10) $\Omega(X)|_{X\to\Gamma} > 0$. $\mathrm{Re}\,\Omega$ vanishes at $\mathrm{Im}\,(XY) = 2tp(X)$. There are a total of seven anti-Stokes lines connected at branch points. They are transcendental and computed numerically, in Fig.2.

Among the seven anti-Stokes lines, only three lines Γ_3, Γ_2 and Γ_1 obey the second condition (10). Line Γ_1 is the boundary of the finger. Lines Γ_3, Γ_2 are shocks.

Selection works as follows. A remote part of the finger ($\arg X = \pi$, $|X| \to \infty$) is not affected by the transition. This selects a branch of $\Omega \sim \frac{4}{5}X^{5/2}$ such that on the upper side of Γ_1, $\mathrm{Re}\,\Omega > 0$. and therefore in both sectors $\{\mathrm{IV}, \mathrm{III}\}$ where $|\arg X - \pi| < 2\pi/5$ at a large X. Then signs of $\mathrm{Re}\,\Omega$ are opposite on both sides of $\Gamma_4, \Gamma_5, \Gamma_6, \Gamma_7$, and signs are positive on both sides of $\Gamma_1, \Gamma_2, \Gamma_3$, as required by (10).

The Rankine-Hugoniot conditions (12 and (9) give the velocity of shocks. Noticing that $\mathrm{Im}\,XY = \sigma X_\perp$, where $X_\perp$ is a projection of a vector-coordinate of a shock to a direction normal to the shock, we get $V_\perp = \frac{X_\perp}{t}$.

Already scaling yields that shocks push the fluid away faster ($\sim t^{5/4}$) than it is absorbed by the drain $\sim t$. Therefore, the finger retreats $v(e_3) = \dot{e}_3 < 0$, smoothing the tip. Indeed, at the endpoints $\Omega(e_{1,2})$ is purely imaginary. Then by virtue of (9) $\mathrm{Im}\ \Omega(e_{1,2}) = \int_{e_3}^{e_{1,2}} \sigma d\ell$ is a mass deficit accumulated on shocks. Eqs.(15), (17) yield:

$$\int_{e_3}^{e_{1,2}} \sigma d\ell = \frac{4}{5}|\psi(e_{1,2})| = \frac{4}{5}(6.34513\ldots)\, t^{5/4}\ . \quad (19)$$

Since the boundary of the fluid moves towards lower pressure, pressure is positive close to sides of the tip, but remains negative around a distant part of the finger. Therefore, in the fluid there are lines of zero pressure (the bright dotted line in Fig.2). They emanate from the end points crossing Γ_1 normally at a point $x = -\eta t^{1/2}$, where velocity vanishes, Fig.2.

Finally, we list angles of zero-pressure line, the shock line and velocity at e_1. Relative to the real axis, they are respectively 0.235061π, 0.3792582π, and 0.451357π. At e_3 the angle between shocks is $2\pi/3$.

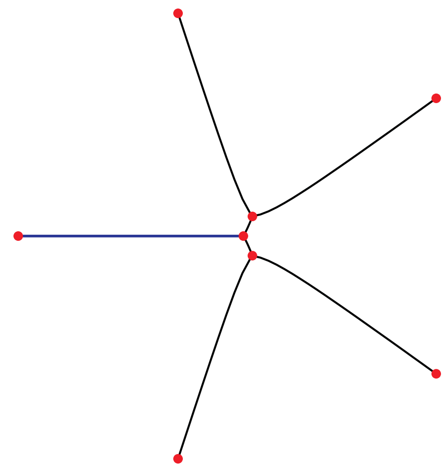


Fig.3. A numerically computed graph of viscous shocks with two generations of branching

10. Branching tree. Evolution through a (2,3)-cusp gives rise to a shock tree. It grows and keeps branching further. An interesting branching tree emerges, Fig.3. We do not know its global structure, but we do know that every generic branching event is locally identical to the transition we just described. It will be interesting to study whether a developed shock's graph with a large number of branches exhibits a scale invariance.

11. Viscous shocks may have different meanings depending on experimental settings. In a Hele-Shaw cell, viscous shocks are narrow channels where an inviscid liquid is compressed and sheared. In a visco-ellastic fluid and in granular materials viscous shocks are cracks, etc. It will be very interesting to see an experimental realization of the branching tree of shocks in viscous flows at small Reynolds numbers.

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