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Two dimensional gravity in genus one in matrix models, topological and liouville approaches

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One-matrix model in p-critical point on torus is considered. The generating function of correlation numbers in genus one is evaluated and used for computation correlation numbers in KdV and CFT frames. It is shown that the correlation numbers in KdV frame in genus one satisfy the Witten topological gravity recurrence relations.

1. Introduction. There exist several different approaches to the 2d Quantum gravity. One of them is the continuous approach. In this approach the theory is determined by the functional integral over all metrics [1]. Calculation of this integral in the conformal gauge leads to the Liouville field theory. Therefore this approach is called the Liouville gravity.

The other way to describe the sum over 2d surfaces is the discrete approach. It is based on the idea of approximation of two-dimensional geometry by an ensemble of planar graphs of big size. Technically the ensemble of graphs is usually defined by expansion into a series of perturbation theory of the integral over $N \times N$ matrices. That is why this approach is called the Matrix Models (further MM). References to the both approaches can be found in the review [2].

After [3, 4, 2] the coincidence of the gravitational dimensions was found the conjecture that both of these approaches describe the same variant of the 2d Quantum gravity appeared. Therefore it was naturally to expect that the correlation numbers will are also the same. However the attempt of a naive identification of the correlation numbers does not lead to the agreement in a general case.

In [5, 6] a conjecture was proposed and checked that there exists a "resonance" transformation of coupling constants in Matrix Models, from the standard definition (the so-called KdV frame) to another (the so-called Liouville or CFT frame), such that new defined correlation numbers of MM coincide with naturally defined ones in the Liouville Gravity.

The form of the transformation was conjectured in [6] for the particular case of the p-critical One-matrix model, which corresponds to the Minimal Liouville gravity $\mathcal{MG}_{2/2p+1}$. The conjectured identity of the correlation numbers was checked up to five-point case in genus zero [6, 7].

At last the third approach – 2d Topological gravity was invented by Witten in [8], who built axiomatics of this theory along the lines of intersection theory. It was conjectured and checked (for genus zero) in [8] that correlation numbers in Topological gravity and in Matrix models coincide. It should be mentioned that this fact takes place if correlation numbers in One-matrix model are calculated in KdV frame.

The article is organized in the following way. At first we review the method of orthogonal polynomials for the solution of Matrix model, the double scaling limit and Douglas string equation. Then we use these tools to compute the torus partition function in p-critical One-matrix model.

We use the explicit expression for the partition function in genus one to compute the correlation numbers in KdV, as well as in CFT frames.

The results in CFT frame should be compared with the correlation numbers in the Minimal Liouville grav-

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ity, which have not been computed yet. In genus one we expect a coincidence similar to that observed in genus zero [5, 6].

After that, we evaluate explicitely the first three correlation numbers in KdV frame. Afterwards we derive the recursion relation in genus one and observe that it coincides with the one of Topological gravity. Then we compute two first correlation numbers in Conformal frame. At the end we discuss some open problems.

2. The method of orthogonal polynomials. Here we give some well-known aspects of the method of orthogonal polynomials [2]. The partition function which includes surfaces of all genera is

$$F(v_k, N) = \log \int dM e^{-\mathrm{tr} V(M)}, \tag{1}$$

where M is hermitian $N \times N$ matrix and $V(M) = N \sum_{k=1}^{p+1} v_k M^{2k}$ is the polynomial potential with different coupling constants v_k, and in further calculation we fix $v_{p+1} = (p+1)!p!/(2p+2)!$ for simplicity. On the other hand, it is known [2] that partition function (1) is expressed through the sum

$$F = \sum_{h=0}^{\infty} N^{2-2h} F_h, \tag{2}$$

where h is genus of surface and F_h is the partition function of all surfaces with genus h. In this paper we evaluate torus partition function F_1, expanding the integral (1) in $1/N$ series.

In this section we carry on evaluation using the method of orthogonal polynomials.

Since the integrand in (1) depends only on the eigenvalues of the matrix M, we can factorize the integration measure into the product of the Haar measure for unitary matrices and an integration measure for eigenvalues.

Thus we have

$$F(v_k, N) = \log \int \prod_{i=1}^{N} d\lambda_i \Delta^2(\lambda) e^{-\sum_i V(\lambda_i)}, \tag{3}$$

where λ_i's are the N eigenvalues of the hermitian matrix M and $\Delta(\lambda) = \prod_{i<j}(\lambda_i - \lambda_j)$ is the Vandermonde determinant. Then we define the set of orthogonal polynomials $P_n(\lambda) = \lambda^n + ...$, by

$$\int_{-\infty}^{\infty} d\lambda e^{-V(\lambda)} P_n(\lambda) P_m(\lambda) = s_n \delta_{nm}. \tag{4}$$

It is easy to see that the Vandermonde determinant can be written as $\Delta(\lambda) = \det P_{j-1}(\lambda_i)$. After some calculations we get the expression for general partition function

$$F = N \sum_{k=1}^{N-1} (1 - k/N) \log(s_k/s_{k-1}). \tag{5}$$

Therefore we need to evaluate the s_k. Due to the assumption that the potential $V(\lambda)$ is even, and also due to orthogonality and normalizing condition of polynomials one can obtain the simple recursion relation

$$\lambda P_k = P_{k+1} + R_k P_{k-1}, \tag{6}$$

where R_k is constant. Then we can derive $\int e^{-V} P_k \lambda P_{k-1} d\lambda = R_k s_{k-1} = s_k$, and thus $s_k/s_{k-1} = R_k$. After that the partition function writes as follows

$$F = N \sum_{k=1}^{N-1} (1 - k/N) \log R_k. \tag{7}$$

The next step is the key relation that will allow us to determine R_k :

$$k s_{k-1} = \int e^{-V} P_k' P_{k-1} = \int e^{-V} V' P_k P_{k-1}. \tag{8}$$

Since the derivative of the polynomial potential $V(\lambda)$ is equal $V'(\lambda) = \sum_{k=1}^{p+1} 2k v_k \lambda^{2k-1}$, we need to evaluate integrals like $\int e^{-V} \lambda^{2n-1} P_k P_{k-1}$.

To do this one should apply relation (6) precisely $2n-1$ times:

$$\lambda^{2n-1} P_k = \lambda^{2n-2}(P_{k+1} + R_k P_{k-1}) = \lambda^{2n-3}(P_{k+2} + R_{k+1}P_k + R_k P_k + R_k R_{k-1} P_{k-2}) = ... = P_{k+2n-1} + ...$$

But from this sum only the terms with P_{k-1} will give contribution to integral (8). The contributions of the such terms to the integral (8) may be visualize paths of $2n-1$ steps ($n-1$ steps up and n steps down) starting at k and ending at $k-1$.

Each step down from m to $m-1$ receives a factor of R_m and each step up receives a factor of unity.

The total number of the paths is given by the binomial coefficient C_{2n-1}^{n} and each path gives a contribution to the factor of s_{k-1} arising due to the integral $\int e^{-V} P_{k-1} P_{k-1}$. Thus for our potential $V(\lambda)$ one can obtain the equation

$$\frac{k}{N} = \tilde{W}(R_k, R_{k\pm1}, ..., R_{k\pm p}), \tag{9}$$

where the function $\tilde{W}$ is the polynomial of R-terms. In explicit form it can be written like

$$\tilde{W}(R_k, R_{k\pm1}, ..., R_{k\pm p}) = \sum_{n=0}^{p+1} 2n v_n \tilde{W}_n = \sum_{n=1}^{p+1} 2n v_n \sum_{\{\sigma_{2n-1}\}} R_{k+m_1} \cdot ... \cdot R_{k+m_n}, \tag{10}$$

where $\{\sigma_{2n-1}\}$ denotes all paths of length $2n-1$ of the type, which is described above. Each path determines the numbers $m_1, ..., m_n$, where $k+m_i$ are the coordinates of points, from which the path goes down. Solving (9) with respect to the R_k's, and putting the answer in (7) we obtain full partition function F. In the next part we will carry on this procedure using perturbation theory with parameter $1/N$.

3. Evaluation of F_0 and F_1. We are going to find $R_k(N)$ which satisfy to equation (10) when N goes to infinity. Therefore we can look for this solution in terms of the smooth function $R(\xi, N)$ of variable $\xi \in [0,1]$, such that $R(\frac{k}{N}, N) = R_k$.

It means that $R(\xi + m/N, N)$ has the Taylor expansion

$$R(\xi + m/N, N) = R(\xi, N) + \\ + \frac{m}{N} R_\xi(\xi, N) + \frac{m^2}{2N^2} R_{\xi\xi}(\xi, N) + O\left(\frac{1}{N^3}\right), \quad (11)$$

where $R_\xi \equiv \partial R/\partial \xi$ and $R_{\xi\xi} \equiv \partial^2 R/\partial \xi^2$.

In addition we assume that the function $R(\xi, N)$ itself can be also expanded in series

$$R(\xi, N) = R(\xi) + \frac{1}{N} R_1(\xi) + \frac{1}{N^2} R_2(\xi) + ..., \quad (12)$$

when N goes to infinity.

Taking into account the first assumption we get from (10) after some calculations (see Appendix B) the following expansion of $\tilde{W}$:

$$\tilde{W}(R_k, R_{k\pm1}, ...) = W(R(\xi, N)) + \frac{1}{N} W_1(R(\xi, N)) + \\ + \frac{1}{N^2} W_2(R(\xi, N)) + O\left(\frac{1}{N^3}\right), \quad (13)$$

where

$$W(R(\xi, N)) = \sum_{n=1}^{p+1} \frac{(2n)!}{n!(n-1)!} v_n R^n(\xi, N), \quad (14)$$

$$W_1(R(\xi, N)) = 0, \quad (15)$$

$$W_2(R(\xi, N)) = \frac{R R_{\xi\xi}}{6} W''(R(\xi, N)) + \frac{R R_\xi^2}{12} W'''(R(\xi, N)), \quad (16)$$

and $W'(R) \equiv dW/dR$, $W'''(R) \equiv d^3W/dR^3$.

Thus the equation (9) takes a form

$$\xi = W(R(\xi, N)) + \frac{R R_{\xi\xi}}{6N^2} W''(R(\xi, N)) + \\ + \frac{R R_\xi^2}{12N^2} W'''(R(\xi, N)) + O\left(\frac{1}{N^4}\right), \quad (17)$$

where $\xi = k/N$.

Using the expansion (12) we have from (17) at $N \to \infty$

$$\xi = W(R(\xi)), \quad (18)$$

$$R_1(\xi) = 0,$$

$$R_2(\xi) = -\frac{R(\xi)}{12W'(R(\xi))} \left(2R_{\xi\xi} W''(R(\xi)) + R_\xi^2 W'''(R(\xi))\right). \quad (19)$$

Let us come back to partition function (7). In order to obtain the coefficients F_h from (2) we need to expand the sum in formula (7) in $1/N$ series. We pass from discrete sum of number k to integral of continious variable ξ. For this procedure one need to use Euler–Maclaurin formula (see Appendix A).

$$F = N^2 \int_0^1 d\xi (1-\xi) \log R(\xi, N) - \frac{N}{2}(G(1) - G(1/N)) + \\ + \frac{1}{12}(G'(1) - G'(1/N)) + O(1/N), \quad (20)$$

where $G(\xi) = (1-\xi)\log R(\xi, N)$. Using the formulae (12) and (19) we obtain from (20)

$$F_0 = \int_0^1 d\xi (1-\xi) \log R, \quad (21)$$

$$F_1 = -\frac{1}{12} \int_0^1 d\xi (1-\xi) \frac{2R_{\xi\xi} W''(R) + R_\xi^2 W'''(R)}{W'(R)}, \quad (22)$$

where $R = R(\xi)$ is the solution of equation (18).

In (21), (22) we keep only first integral term in r.h.s. of (20) and omit the others. The reason is that in the vicinity of critical point, which we will be interesting below, these terms have less singularity than the integral.

4. The vicinity of p-critical point. The p-critical point are defined by the system of equations

$$W(R_c) = 1, \quad W'(R_c) = 0, \quad ... \quad W^{(p)}(R_c) = 0. \quad (23)$$

Actually it is the system of equations, which determine coefficients v_k^c, $k = 1, ..., p$, and define the R_c. Thus if we put the $v_k = v_k^c$, then we have from (18)

$$\xi = 1 + (R - R_c)^{p+1}. \quad (24)$$

Then if we consider the special small deviations $\delta v_k = v_k - v_k^c$, such that equations in (23) take a form

$$W(R_c) = 1 + t_{p-1}, \quad W'(R_c) = t_{p-2}, \quad ... \\ W^{(p-1)}(R_c) = t_0, \quad W^{(p)}(R_c) = 0. \quad (25)$$

Denoting $u = R - R_c$ one can obtain from (18)

$$\xi = W(u) = u^{p+1} + t_0 u^{p-1} + \sum_{k=1}^{p-1} t_k u^{p-k-1} + 1. \quad (26)$$

We can rewrite (26), making a substitution $\xi = 1 - y$, as follows

$$\mathcal{P}(u) + y = 0, \quad (27)$$

where

$$\mathcal{P}(u) \stackrel{\text{def}}{=} u^{p+1} + t_0 u^{p-1} + \sum_{k=1}^{p-1} t_k u^{p-k-1}. \quad (28)$$

In view of formula (27), in further we treat the variable u, as the function of variables $\{t_k\}$ and y: $u = u(\{t_k\}, y)$ (or $u(y)$ for short).

Therefore from (21) and (22) one can get for singular part of the partition function [2] of genus zero and genus one

$$F_0 = \frac{1}{R_c} \int_0^1 dy\, y\, u(y), \quad (29)$$

$$F_1 = -\frac{1}{12} \int_0^1 dy\, y \left(\frac{2\mathcal{P}''(u) u_{yy} + \mathcal{P}'''(u) u_y^2}{\mathcal{P}'(u)} \right), \quad (30)$$

where

$$\mathcal{P}'(u) \equiv \frac{d\mathcal{P}}{du}, \quad \mathcal{P}''(u) \equiv \frac{d^2\mathcal{P}}{du^2}, \quad \mathcal{P}'''(u) \equiv \frac{d^3\mathcal{P}}{du^3}.$$

The singularity of F_1 arises due to the contribution of the vicinity of the point $y = 0$ in the integral.

Using (28) in relations $u_y = -1/\mathcal{P}'$ and $u_{yy} = -\mathcal{P}''/(\mathcal{P}')^3$, we obtain the following final answers for the singular parts of the partition functions F_0 and F_1

$$F_0 = \frac{1}{2} \int_0^{u^*} \mathcal{P}^2(u) du, \quad (31)$$

$$F_1 = -\frac{\log \mathcal{P}'(u^*)}{12}, \quad (32)$$

where $u^* = u^*(t_0, t_1, ..., t_{p-1})$ is the suitably chosen root [6] of the polynomial $\mathcal{P}(u)$. The result (32) was obtained by different method in [9].

In further we want to evaluate correlation numbers in the critical point $(t_0, 0, ..., 0)$. Let us denote the cosmological constant $t_0 = \mu$. The deviation from critical point is defined by parameters t_k, $k = 1, .., p-1$. In the vicinity of the critical point $\mu \ll 1$, $t_k \sim \mu^{\frac{k+2}{2}}$, the scaling partition function, which corresponds to the surfaces of genus h, is scale invariant

$$F_h[\lambda\mu, \lambda^{\frac{k+2}{2}} t_k] = (\lambda^{p+3/2})^{1-h} F_h[\mu, t_k]. \quad (33)$$

Double scaling limit corresponds to $N \to \infty$, while μ and $t_k \to 0$ proportionally $(N^2\varepsilon^2)^{-\frac{2}{2p+3}}$ and $(N^2\varepsilon^2)^{-\frac{k+2}{2p+3}}$ correspondingly, where ε is some finite parameter.

Making suitable replacement of variables, using the rescaling (33) and performing the substitution $F/N^2\varepsilon^2 \to F$ for simplicity, we arrive to the expression for the partition function in the double scaling limit $F[\mu, t_k, \varepsilon]$

$$F[\mu, t_k, \varepsilon] = \sum_{h=0}^{\infty} \varepsilon^{2h} F_h[\mu, t_k], \quad (34)$$

where ε is the parameter, which is responsible for genus expansion. The expansion (34) is similar to the ones in Liouville gravity.

In the next section we derive the same expression for the torus partition function by method different to ones in section 2. Namely we will use the string equation for the partition function $F[\mu, t_k, \varepsilon]$ and the expansion respect to the small parameter ε.

5. String equation. In this section we show how to obtain the expressions (31) and (32) using the double scaling limit and the Douglas string equation. The string equation is the equation for function $u(x, \varepsilon, \mu, t_k)$ (or $u(x, \varepsilon)$ for short), which is connected with the partition function in the double scaling limit $F[\mu, t_k, \varepsilon]$ as

$$u(x, \varepsilon) = \frac{d^2 F}{dx^2}, \quad (35)$$

and looks as follows

$$[\hat{P}, \hat{Q}] = 1, \quad (36)$$

where $\hat{Q} = \varepsilon^2 d^2 + u(x)$ and $\hat{P} = -\sum_{k=1}^{p+1} t_{p-1-k} \hat{Q}_+^{k-1/2}$ are two differential operators.

$\hat{Q}_+^{k-1/2}$ is the non-negative part of the pseudo-differential operator $\hat{Q}^{k-1/2}$.

In view of (34) we look for $u(x)$ in the form

$$u(x, \varepsilon) = \sum_{h=0}^{\infty} \varepsilon^{2h} u_h(x), \quad (37)$$

where, obviously, u_h

$$u_h(x) = \frac{d^2 F_h}{dx^2}. \quad (38)$$

It is known [2], that

$$[\hat{Q}_+^{k-1/2}, \hat{Q}] = \frac{dS_k}{dx}, \quad (39)$$

where the coefficients $S_k(u)$ obey the recursion relation

$$\frac{dS_{k+1}}{dx} = u\frac{dS_k}{dx} + \frac{1}{2} u_x S_k + \frac{\varepsilon^2}{4} \frac{d^3 S_k}{dx^3}, \quad (40)$$

with the boundary conditions $S_0 = \frac{1}{2}$ and $S_k (k \neq 0)$ vanish at $u = 0$, and we assume that $u_x = \partial u/\partial x$, $u_{xx} = \partial^2 u/\partial x^2$. Thus we obtain from (36) the relation

$$\sum_{k=1}^{p+1} t_{p-1-k} S_k(u) = -x. \tag{41}$$

The solution of the recursion relations (40) (see Appendix), including the first three terms is

$$S_k(u) = \frac{C_{2k}^k}{2^{2k+1}} \left(u^k + \frac{\varepsilon^2 k(k-1)}{6} u^{k-2} u_{xx} + \right. \\ \left. + \frac{\varepsilon^2 k(k-1)(k-2)}{12} u^{k-3} u_x^2 \right) + O(\varepsilon^4). \tag{42}$$

Thus after rescaling the parameter $t_k \to \frac{2^{2k+1}}{C_{2k}^k} t_k$, we can obtain from (41) that

$$\mathcal{P}(u) + \varepsilon^2 \left(\frac{1}{6} \mathcal{P}''(u) u_{xx} + \frac{1}{12} \mathcal{P}'''(u) u_x^2 \right) = O(\varepsilon^4), \tag{43}$$

where $\mathcal{P}(u)$ is the polynomial from (28) and $x = t_{p-1}$, $t_{-2} = 1$, $t_{-1} = 0$.

Using the expansion (37), we get from (43)to the zeroth order in the ε, that $u_0(x)$ obeys

$$\mathcal{P}(u_0) = 0, \tag{44}$$

therefore

$$u_0 = u^*(t_1, ..., t_{p-2}, x), \tag{45}$$

where u^* is the suitably chosen root of the polynomial $\mathcal{P}(u)$. To the second order in the ε gives for the $u_1(t_1, ..., t_{p-2}, x)$ the following expression

$$u_1 = -\frac{\mathcal{P}'''(u^*)(u_x^*)^2 + 2\mathcal{P}''(u^*) u_{xx}^*}{12 \mathcal{P}'(u^*)}. \tag{46}$$

Knowing u_0 and u_1 we can find corresponding the partition functions F_0 and F_1, using (38) and the fact that if F and u^* are connected by relation

$$\frac{\partial^2 F}{\partial x^2} = f(u^*), \tag{47}$$

then

$$F = -\int_0^{u^*} \mathcal{P}(u) \mathcal{P}'(u) f(u) du. \tag{48}$$

This formula can be checked by straightforward calculation.

Integrating by parts and omitting the regular terms, we get from (38), (45), (46) and (48)

$$F_0 = \frac{1}{2} \int_0^{u^*} \mathcal{P}^2(u) du, \tag{49}$$

$$F_1 = -\frac{\log \mathcal{P}'(u^*)}{12}. \tag{50}$$

We see that these formulae coincide with (31), (32). In the next sections we will use this formula for the torus partition function in order to obtain expressions for correlation numbers.

6. Evaluation of correlation numbers in genus one in KdV frame. In the scaling limit near the p-critical point the partition function of the One-matrix model on torus can be described in terms of the solution of the "string equation"

$$\mathcal{P}(u) = 0, \tag{51}$$

where $\mathcal{P}(u)$ is the polynomial of degree $p+1$ (p is natural number)

$$\mathcal{P}(u) = u^{p+1} + t_0 u^{p-1} + \sum_{k=1}^{p-1} t_k u^{p-k-1}, \tag{52}$$

with the parameters t_k controlling the deviation from the p-critical point.The singular part of the partition function on torus in the matrix models $F_1(t_0, t_1, ... t_{p-1})$ can be described according to (50), as

$$F_1 = -\frac{\log \mathcal{P}'(u^*)}{12}, \tag{53}$$

where $u^* = u^*(t_0, t_1, ..., t_{p-1})$ is the suitably chosen root of the polynomial (52). Also introduce $u_c = u^*(t_0, 0, ..., 0) = \sqrt{-t_0}$. The correlation numbers are expressed through the formula

$$\langle O_{k_1} ... O_{k_n} \rangle_1 = \left. \frac{\partial^n F_1}{\partial t_{k_1} ... \partial t_{k_n}} \right|_{t_1 = ... = t_{p-1} = 0}, \tag{54}$$

where the index $\langle \ \rangle_1$ denotes the correlation numbers on torus. The first three correlation numbers are (see Appendix D)

$$\langle O_k \rangle_1 = \frac{p+k}{24} u_c^{-k-2},$$

$$\langle O_{k_1} O_{k_2} \rangle_1 =$$

$$= \frac{(p+2+k_1+k_2)(k_1+k_2) + 2p - k_1 k_2}{48} u_c^{-k_1-k_2-4},$$

$$\langle O_{k_1} O_{k_2} O_{k_3} \rangle_1 = \frac{1}{96} \left(\frac{2k^3}{3} + \frac{k_i^3}{3} + (p+4)k^2 + 2k_i^2 + \right. \\ \left. + (6p+8)k + 8p - 2k_1 k_2 k_3 \right) u_c^{-k_1-k_2-k_3-6}, \tag{55}$$

where $k = k_1 + k_2 + k_3$, $k_i^2 = k_1^2 + k_2^2 + k_3^2$, and $k_i^3 = k_1^3 + k_2^3 + k_3^3$.

7. Comparison with Topological Gravity. In paper [8] E.Witten gave the definition of 2d Topological gravity. The recursion relation between correlation numbers has been derived in [8] by studying intersection theory. Using this relation Witten computed correlation numbers in genus zero and checked their coincidence with expressions for correlation numbers in One-matrix model. This resemblance lead Witten to the conjecture about the equivalence between Topological gravity and One-matrix model. Witten in [12] proved his conjecture using results of M.Kontsevich [13].

In this section we show that the same recursion relation in genus one as well as in genus zero holds also in One-matrix model. Our approach uses the explicit expression for the partition function of One-matrix model and differs from that used by Kontsevich.

Let $\langle\,\rangle_0$ and $\langle\,\rangle_1$ denote the genus zero and genus one correlation numbers. The recursion relations between correlation numbers from [8] look as follows

$$\langle\sigma_{k_1}\sigma_{k_2}...\sigma_{k_s}\rangle_0 = \\ =k_1\sum_{S=X\cup Y}\langle\sigma_{k_1-1}\prod_{i\in X}\sigma_{k_i}\sigma_0\rangle_0\langle\sigma_0\prod_{j\in Y}\sigma_{k_j}\sigma_{k_{s-1}}\sigma_{k_s}\rangle_0, \quad (56)$$

$$\langle\sigma_{k_1}\sigma_{k_2}...\sigma_{k_s}\rangle_1 = \\ = \frac{1}{12}k_1\langle\sigma_{k_1-1}\sigma_{k_2}...\sigma_{k_s}\sigma_0\sigma_0\rangle_0 + \\ + k_1\sum_{S=X\cup Y}\langle\sigma_{k_1-1}\prod_{i\in X}\sigma_{k_i}\sigma_0\rangle_0\langle\sigma_0\prod_{j\in Y}\sigma_{k_j}\rangle_1, \quad (57)$$

where $\sigma_k \leftrightarrow O_{p-k-1}$ and the symbol $\sum_{S=X\cup Y}$ represent a sum over all decomposition of $S=\{2,3,...,s\}$ as a union of the two sets X and Y.

Witten has shown that the relations (56) and (57) are fulfilled for correlation numbers $\langle\sigma_{k_1}...\sigma_{k_s}\rangle_{0,1}$ of the more general theory which depends on relevant parameters $\{a_k\}$ ($a_k \leftrightarrow t_{p-k-1}$) in such way that expectation values of any observable N obey

$$\frac{\partial}{\partial a_k}\langle N\rangle = \langle\sigma_k N\rangle, \quad (58)$$

if the following relations

$$\langle\sigma_{k_1}\sigma_{k_2}\sigma_{k_3}\rangle_0 = k_1\langle\sigma_{k_1-1}\sigma_0\rangle_0\langle\sigma_0\sigma_{k_2}\sigma_{k_3}\rangle_0, \quad (59)$$

$$\langle\sigma_k\rangle_1 = \frac{1}{12}k\langle\sigma_{k-1}\sigma_0\sigma_0\rangle_0 + k\langle\sigma_{k-1}\sigma_0\rangle_0\langle\sigma_0\rangle_1 \quad (60)$$

hold.

In One-matrix model the correlation numbers with arbitrary $\{t_k\}$ are given by the formula

$$\langle O_{k_1}...O_{k_n}\rangle_{0,1} = \frac{\partial^n F_{0,1}}{\partial t_{k_1}...\partial t_{k_n}}. \quad (61)$$

Therefore the assumption (58) is holds automatically. The fulfillment of (59) for correlation numbers in genus zero was checked by A.B. Zamolodchikov [10].

Below we check (59) and (60) using explicit expressions for F_0 and F_1 from (49) and (50). In terms of observables O_k in One-matrix model, the expressions (59) and (60) writes ($\sigma_k \leftrightarrow O_{p-k-1}$)

$$\langle O_{p-k_1-1}O_{p-k_2-1}O_{p-k_3-1}\rangle_0 = \\ = k_1\langle O_{p-k_1}O_{p-1}\rangle_0\langle O_{p-1}O_{p-k_2-1}O_{p-k_3-1}\rangle_0, \quad (62)$$

$$\langle O_{p-k-1}\rangle_1 = \\ =\frac{1}{12}k\langle O_{p-k}O_{p-1}O_{p-1}\rangle_0 + k\langle O_{p-k}O_{p-1}\rangle_0\langle O_{p-1}\rangle_1. \quad (63)$$

From formula (49) at arbitrary $\{t_k\}$ one can get

$$\langle O_{k_1}O_{k_2}\rangle_0 = \frac{\partial^2 F_0}{\partial t_{k_1}\partial t_{k_2}} = \frac{(u^*)^{2p-k_1-k_2-1}}{2p-k_1-k_2-1}, \\ \langle O_{k_1}O_{k_2}O_{k_3}\rangle_0 = \frac{\partial^3 F_0}{\partial t_{k_1}\partial t_{k_2}\partial t_{k_3}} = -\frac{(u^*)^{3p-k_1-k_2-k_3-3}}{\mathcal{P}'(u^*)}. \quad (64)$$

Thus it is easy to see from (64) that equality (62) is fulfilled. From the torus partition function (50) also at arbitrary $\{t_k\}$ we get

$$\langle O_k\rangle_1 = \frac{\partial F_1}{\partial t_k} = \\ = -\frac{p-k-1}{12\mathcal{P}'(u^*)}(u^*)^{p-k-2} + \frac{\mathcal{P}''(u^*)}{12(\mathcal{P}'(u^*))^2}(u^*)^{p-k-1}. \quad (65)$$

The expressions (64) and (65) indeed satisfy the equation (63). Consequently we have proved that the correlation numbers in One-matrix model in KdV frame satisfy the recurrence relation (56) and (57), assuming replacement $\sigma_k \to O_{p-k-1}$.

8. Evaluation of correlation numbers in the CFT frame. The CFT frame is defined by a different set of parameters $\{\lambda_k\}$, which are associated with $\{t_k\}$ by "resonance" transformation [6]. As it was shown in [6] after "resonance" transformation the polynomial $\mathcal{P}(u,\{t_k\})$ from (52) up to the factor $\frac{(p+1)!}{(2p-1)!!}u_c^{p+1}$ takes the form

$$Q(x,\{\lambda_k\})=\sum_{n=0}^{\infty}\sum_{k_1...k_n=1}^{p-1}\frac{\lambda_{k_1}...\lambda_{k_n}}{n!}\frac{d^{n-1}}{dx^{n-1}}L_{p-\sum k_i-n}(x), \quad (66)$$

where $x=u/u_c$, u_c is u^* at $\lambda_1,...,\lambda_{p-1}=0$ and $L_n(x)$ are the Legendre polynomials. We also assume that

$\left(\frac{d}{dx}\right)^{-1} L_p = \int L_p dx = \frac{L_{p+1}-L_{p-1}}{2p+1}$. Below we use the notation

$$Q_{k_1\ldots k_n}(x) = \frac{d^{n-1}}{dx^{n-1}} L_{p-\sum k_i - n}(x),$$
$$Q_0(x) = \frac{L_{p+1}-L_{p-1}}{2p+1}. \tag{67}$$

The correlation numbers are expressed through the formula

$$\langle \mathcal{O}_{k_1} \ldots \mathcal{O}_{k_n} \rangle_1 = \left. \frac{\partial^n F_1}{\partial \lambda_{k_1} \ldots \partial \lambda_{k_n}} \right|_{\lambda_1 = \ldots = \lambda_{p-1} = 0}. \tag{68}$$

If we calculated correlation numbers for the partition function in genus zero, inserting the polynomial $Q(x, \{\lambda_k\})$ instead of $P(x, t_k)$ in (49) and (50) we would obtained results in [6].

Let us compute the partition function in genus one. Thus taking into account the common formulas for correlation numbers (96) and (97) in Appendix D and using some values for Legendre polynomials and consequently for polynomial $Q(x, \{\lambda_k\})$ in critical point ($x = 1$)

$$Q'(1) = 1, \quad Q''(1) = \frac{p(p+1)}{2},$$
$$Q'''(1) = \frac{(p+2)(p+1)p(p-1)}{8}, \quad Q_{k_i} = 1,$$
$$Q'_{k_i}(1) = \frac{(p-k_i)(p-k_i-1)}{2},$$
$$Q''_{k_i}(1) = \frac{1}{8}\prod_{r=1}^{4}(p-r-k_i+2),$$
$$Q'_{k_i k_j}(1) = \frac{1}{8}\prod_{r=1}^{4}(p-r-k_i-k_j+1), \tag{69}$$

one can obtain from (68) the first two correlation numbers of the partition function in genus one in CFT frame

$$\langle \mathcal{O}_k \rangle_1 = \frac{(2p-k)(k+1)}{24},$$
$$\langle \mathcal{O}_{k_1} \mathcal{O}_{k_2} \rangle_1 = -\frac{1}{24}(1+k_1)(1+k_2) \times$$
$$\times ((k_1+k_2-2p+2)(k_1+k_2) - k_1 k_2 - 4p). \tag{70}$$

9. Conclusion. In this paper we have derived the torus partition function F_1 in p-critical One-matrix model. Using the explicit expression for the partition function in genus one we compute the correlation numbers in KdV, as well as in CFT frames.

We show the fulfillment of recurrence relation for correlation numbers in One-matrix model in KdV frame in genus one, which are the same as that in 2d Topological gravity.

The results in CFT frame should be compared against the correlation numbers in the Minimal Liouville gravity, which have not been computed yet. We expect the coincidence in genus one similarly that was observed on sphere [5, 6, 14] and on disk [15].

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Appendix

A. For our aim in section 3 we use Euler–Maclaurin formula [11]. It helps express summation of discrete function through integration of this function and some other terms.

Let function $f(x)$ is considered in section $[a, b]$. Let $h = (b-a)/n$, where n is natural number, then

$$\sum_{k=1}^{n} f(a+(k-1)h) = \frac{1}{h}\int_a^b f(x)dx - \frac{1}{2}(f(b)-f(a)) +$$
$$+ \sum_{m=1}^{\infty} h^{2m-1}\frac{B_{2m}}{(2m)!}(f^{(2m-1)}(b) - f^{(2m-1)}(a)), \tag{71}$$

where B_m are Bernoulli numbers ($B_0 = 1$, $B_1 = -1/2$, $B_2 = 1/6$). In our analysis of torus partition function we only use terms in Euler–Maclaurin formula up to $m = 1$.

B. In this section we deal with part of the sum from (10)

$$\tilde{W}_n = 2nv_n \sum_{\{\sigma_{2n-1}\}} R_{k+m_1} \cdot \ldots \cdot R_{k+m_n}, \tag{72}$$

where $\{\sigma_{2n-1}\}$ denotes all paths of $2n-1$ steps ($n-1$ steps up and n steps down) starting at k and ending at $k-1$. Each step down from m to $m-1$ receives a factor of R_m and each step up receives a factor of unity.

We assume the existence of smooth function $R(\xi, N)$ of variable $\xi \in [0,1]$, such that $R(k/N, N) = R_k$. Thus $R(\xi + m/N, N)$ and $\tilde{W}_n$ have the Taylor expansion

$$R(\xi + m/N, N) = R(\xi, N) +$$
$$+ \frac{m}{N}R_\xi(\xi, N) + \frac{m^2}{2N^2}R_{\xi\xi}(\xi, N) + O\left(\frac{1}{N^3}\right), \tag{73}$$

$$\tilde{W}_n = W_n + \frac{1}{N}W_{1n} + \frac{1}{N^2}W_{2n} + O\left(\frac{1}{N^3}\right). \tag{74}$$

Since

$$R(\xi+\frac{m_1}{N},N)\cdot\ldots\cdot R(\xi+\frac{m_n}{N},N)=$$
$$=R^n+\frac{R^{n-1}R_\xi}{N}\sum_{i=1}^n m_i+\frac{R^{n-1}R_{\xi\xi}}{2N^2}\sum_{i=1}^n m_i^2+$$
$$+\frac{R^{n-2}R_\xi^2}{N^2}\sum_{i<j}^n m_i m_j. \quad (75)$$

Thus from (74) we obtain

$$W_n=2nv_nR^n\sum_{\{\sigma_{2n-1}\}}1,$$
$$W_{1n}=2nv_nR^{n-1}R_\xi\sum_{\{\sigma_{2n-1}\}}\sum_{i=1}^n m_i,$$
$$W_{2n}=2nv_nR^{n-2}R_\xi^2\sum_{\{\sigma_{2n-1}\}}\sum_{i<j}^n m_im_j+ \quad (76)$$
$$+nv_nR^{n-1}R_{\xi\xi}\sum_{\{\sigma_{2n-1}\}}\sum_{i=1}^n m_i^2.$$

Therefore we need to evaluate the sums

$$P_n=\sum_{\{\sigma_{2n-1}\}}1,\quad M_n=\sum_{\{\sigma_{2n-1}\}}\sum_{i=1}^n m_i,$$
$$A_n=\sum_{\{\sigma_{2n-1}\}}\sum_{i=1}^n m_i^2,\quad B_n=\sum_{\{\sigma_{2n-1}\}}\sum_{i<j}^n m_im_j. \quad (77)$$

First of all, it is easy to see that P_n, which is the total number of paths, is given by the binomial coefficient C^n_{2n-1}.

Thus we have for $P_n=C^n_{2n-1}$, and one can writes

$$P_n=\sum_{\{\sigma_{2n-1}\}}1=C^n_{2n-1},$$
$$M_n=\sum_{\{\sigma_{2n-1}\}}\sum_{i=1}^n m_i=\sum_{x,y=0}^{n-1}(y-x)C^x_{x+y}C^{n-1-x}_{2n-2-x-y}=0,$$
$$A_n=\sum_{\{\sigma_{2n-1}\}}\sum_{i=1}^n m_i^2=\sum_{x,y=0}^{n-1}(y-x)^2C^x_{x+y}C^{n-1-x}_{2n-2-x-y}=$$
$$=\frac{n(n-1)}{3}C^n_{2n-1}, \quad (78)$$
$$B_n=\sum_{\{\sigma_{2n-1}\}}\sum_{i<j}^n m_im_j=$$
$$=\sum_{x,y=0}^{n-1}(y-x)C^{n-1-x}_{2n-2-x-y}\times$$
$$\times\sum_{x_1=0}^{x-1}\sum_{y_1=0}^{y}(y_1-x_1)C^{x_1}_{x_1+y_1}C^{y-y_1}_{x-1+y-y_1-x_1}=$$
$$=\frac{n(n-1)(n-2)}{12}C^n_{2n-1}.$$

Thus we obtain for $W=\sum_{n=0}^{p+1}2nv_nW_n$, $W_1=\sum_{n=0}^{p+1}2nv_nW_{1n}$ and for $W_2=\sum_{n=0}^{p+1}2nv_nW_{2n}$, following expressions

$$W(R(\xi,N))=\sum_{n=1}^{p+1}\frac{(2n)!}{n!(n-1)!}v_nR^n(\xi,N),$$
$$W_1(R(\xi,N))=0, \quad (79)$$
$$W_2(R(\xi,N))=\frac{RR_{\xi\xi}}{6}W''(R(\xi,N))+\frac{RR_\xi^2}{12}W'''(R(\xi,N)).$$

C. In this section we solve the recurrence relation for $S_k[u,u',u'',...]$

$$\frac{dS_{k+1}}{dx}=u\frac{dS_k}{dx}+\frac{1}{2}u_xS_k+\frac{\varepsilon^2}{4}\frac{d^3S_k}{dx^3}, \quad (80)$$

with the boundary conditions $S_0=\frac{1}{2}$ and $S_k(k\neq 0)$ vanish at $u=0$. From the form of equation (80) it follows that S_k is expanded into series of ε:

$$S_k=S_k^0+\varepsilon S_k^1+...+\varepsilon^lS_k^l+.... \quad (81)$$

and l-th term in this expansion contains the common number of derivatives equal to l. In order to obtain the partition function of genus one we can limit this expansion up to the first three terms, thus we have

$$S_k=P_k(u)+\varepsilon M_k(u)u_x+$$
$$+\varepsilon^2(A_k(u)u_{xx}+B_k(u)u_x^2)+O(\varepsilon^3). \quad (82)$$

For the r.h.s. of the recurrence relation (80) we obtain

$$u\frac{dS_k}{dx}=P'_kuu_x+\varepsilon\left(M'_kuu_x^2+M_kuu_{xx}\right)+$$
$$+\varepsilon^2\left(A_kuu_{xxx}+B'_kuu_x^3+(2B_k+A'_k)uu_xu_{xx}\right)+$$
$$+O(\varepsilon^3),$$
$$\frac{1}{2}u_xS_k=\frac{1}{2}P_ku_x+\varepsilon\frac{M_k}{2}u_x^2+$$
$$+\varepsilon^2\left(\frac{A_k}{2}u_xu_{xx}+\frac{B_k}{2}u_x^3\right)+O(\varepsilon^3),$$
$$\frac{\varepsilon^2}{4}\frac{d^3S_k}{dx^3}=\varepsilon^2\left(\frac{1}{4}P'''_ku_x^3+\frac{3}{4}P''_ku_xu_{xx}+\frac{1}{4}P'_ku_{xxx}\right)+$$
$$+O(\varepsilon^3), \quad (83)$$

and for the l.h.s

$$\frac{dS_{k+1}}{dx}=P'_{k+1}u_x+\varepsilon\left(M'_{k+1}u_x^2+M_{k+1}u_{xx}\right)+$$
$$+\varepsilon^2\left(A_{k+1}u_{xxx}+B'_{k+1}u_x^3+\left(2B_{k+1}+A'_{k+1}\right)u_xu_{xx}\right)+$$
$$+O(\varepsilon^3). \quad (84)$$

In the order ε^0 we have one equation

$$P'_{k+1}=P'_ku+\frac{1}{2}P_k, \quad (85)$$

with boundary condition $P_0 = \frac{1}{2}$. And if we make substitution $P_k = p_k u^k$, where p_k is constant, we obtain recurrence relation for p_k

$$p_{k+1} = \frac{2k+1}{2(k+1)} p_k. \tag{86}$$

Solving this equation we find that $p_k = C_{2k}^k / 2^{2k+1}$, where $C_{2k}^k = (2k)!/k!k!$ is the binomial coefficient. Thus we derive that $P_k(u) = \left(C_{2k}^k / 2^{2k+1}\right) u^k$.

For order ε we have two equations

$$\begin{gathered} M'_{k+1} = M'_k u + \frac{1}{2} M_k, \\ M_{k+1} = M_k u, \end{gathered} \tag{87}$$

with boundary condition $M_0 = 0$. The system of equations (87) has only one solution $M_k = 0$. It easy to see that already from $M_k = 0$ it follows, that all terms of odd order in ε will be qual to zero.

In order ε^2 we have system of three equations

$$\begin{gathered} A_{k+1} = A_k u + \frac{1}{4} P'_k, \\ B'_{k+1} = B'_k u + \tfrac{1}{2} B_k + \frac{1}{4} P'''_k, \\ 2B_{k+1} + A'_{k+1} = (2B_k + A'_k)\, u + \frac{1}{2} A_k + \tfrac{3}{4} P''_k, \end{gathered} \tag{88}$$

with boundary conditions $A_0 = B_0 = 0$. Let us solve the first one.

$$A_{k+1} = A_k u + \frac{1}{4} P'_k. \tag{89}$$

In order to get rid of the heterogeneous component $\frac{1}{4}\frac{\partial P_k}{\partial u}$, we will look for A_k in form $A_k = a_k \frac{k(k-1)}{4}\left(\frac{C_{2k}^k}{2^{2k+1}}\right) u^{k-2}$, where a_k are constants, and we obtain for a_k from (89)

$$a_{k+1}(k + 1/2) - a_k(k-1) = 1. \tag{90}$$

Solution for this equation is $a_k = \frac{2}{3}$, thus we have $A_k = \frac{k(k-1)}{6}\left(\frac{C_{2k}^k}{2^{2k+1}}\right) u^{k-2}$. For the second equation from (88) we put $B_k = b_k \frac{k(k-1)(k-2)}{4}\left(\frac{C_{2k}^k}{2^{2k+1}}\right) u^{k-3}$, then we derive equation for the constants b_k:

$$b_{k+1}(k + 1/2) - b_k(k - 5/2) = 1, \tag{91}$$

and obtain $b_k = \frac{1}{3}$, therefore $B_k = \frac{k(k-1)(k-2)}{12} \times \left(\frac{C_{2k}^k}{2^{2k+1}}\right) u^{k-3}$. We can see that these solutions for A_k and B_k satisfy the third equation in (88).

Summarizing all results, we have

$$\begin{aligned} P_k &= \left(\frac{C_{2k}^k}{2^{2k+1}}\right) u^k, \quad M_k = 0, \\ A_k &= \frac{k(k-1)}{6}\left(\frac{C_{2k}^k}{2^{2k+1}}\right) u^{k-2} = \frac{1}{6} P''_k \\ B_k &= \frac{k(k-1)(k-2)}{12}\left(\frac{C_{2k}^k}{2^{2k+1}}\right) u^{k-3} = \frac{1}{12} P'''_k, \end{aligned} \tag{92}$$

therefore one can writes S_k as follows

$$S_k = P_k + \varepsilon^2 \left(\frac{1}{6} P''_k u_{xx} + \frac{1}{12} P'''_k u_x^2\right) + O(\varepsilon^4). \tag{93}$$

D. The singular part of the partition function on torus $F_1(t_0, t_1, \ldots t_{p-1})$ is

$$F_1 = -\frac{\log \mathcal{P}'(u^*)}{12}, \tag{94}$$

where $u^* = u^*(t_0, t_1, \ldots, t_{p-1})$ is the suitably chosen root of the polynomial

$$\mathcal{P}(u) = u^{p+1} + t_0 u^{p-1} + \sum_{k=1}^{p-1} t_k u^{p-k-1}. \tag{95}$$

The correlation numbers are expressed through the formula (54). Thus in common form first two correlation numbers are (denote $\mathcal{P}_k = \partial\mathcal{P}/\partial t_k$)

$$\langle O_{k_1} \rangle_1 = -\frac{1}{12}\left(\frac{\mathcal{P}'_{k_1}}{\mathcal{P}'} - \frac{\mathcal{P}''\mathcal{P}_{k_1}}{(\mathcal{P}')^2}\right), \tag{96}$$

$$\begin{gathered} \langle O_{k_1} O_{k_2} \rangle_1 = -\frac{1}{12} \times \\ \times \left(\frac{\mathcal{P}'_{k_1 k_2}}{\mathcal{P}'} - \frac{\mathcal{P}''_{k_1}\mathcal{P}_{k_2} + \mathcal{P}''_{k_2}\mathcal{P}_{k_1} + \mathcal{P}'_{k_1}\mathcal{P}'_{k_2} + \mathcal{P}''\mathcal{P}_{k_1 k_2}}{(\mathcal{P}')^2} + \right. \\ + \frac{2\mathcal{P}''\mathcal{P}'_{k_1}\mathcal{P}_{k_2} + 2\mathcal{P}''\mathcal{P}'_{k_2}\mathcal{P}_{k_1} + \mathcal{P}'''\mathcal{P}_{k_1}\mathcal{P}_{k_2}}{(\mathcal{P}')^3} - \\ \left. - \frac{2(\mathcal{P}'')^2\mathcal{P}_{k_1}\mathcal{P}_{k_2}}{(\mathcal{P}')^4}\right). \end{gathered} \tag{97}$$

And the third correlation number is (denote $\mathcal{P}_i = \mathcal{P}_{k_i} = \partial\mathcal{P}/\partial k_i$)

$$\langle O_{k_1}O_{k_2}O_{k_3}\rangle_1 = -\frac{1}{12}\times$$
$$\times\left(\frac{\mathcal{P}'_{123}}{\mathcal{P}'}-\frac{\mathcal{P}''_{(12}\mathcal{P}_{3)}+\mathcal{P}'_{(12}\mathcal{P}_{3)}+\mathcal{P}''_{(1}\mathcal{P}_{23)}+\mathcal{P}''\mathcal{P}_{123}}{(\mathcal{P}')^2}+\right.$$
$$+\frac{1}{(\mathcal{P}')^3}\Big(2\mathcal{P}''\mathcal{P}'_{(12}\mathcal{P}_{3)}+\mathcal{P}'''_{(1}\mathcal{P}_2\mathcal{P}_{3)}+2\mathcal{P}''_{(1}\mathcal{P}'_2\mathcal{P}_{3)}+$$
$$+\mathcal{P}'''\mathcal{P}_{(12}\mathcal{P}_{3)}+2\mathcal{P}''\mathcal{P}'_{(1}\mathcal{P}_{23)}+2\mathcal{P}'_1\mathcal{P}'_2\mathcal{P}'_3\Big)-\frac{1}{(\mathcal{P}')^4}\times$$
$$\times\Big(4\mathcal{P}''\mathcal{P}''_{(1}\mathcal{P}_2\mathcal{P}_{3)}+6\mathcal{P}''\mathcal{P}'_{(1}\mathcal{P}'_2\mathcal{P}_{3)}+2(\mathcal{P}'')^2\mathcal{P}_{(12}\mathcal{P}_{3)}+$$
$$+3\mathcal{P}'''\mathcal{P}'_{(1}\mathcal{P}_2\mathcal{P}_{3)}+\mathcal{P}''''\mathcal{P}_1\mathcal{P}_2\mathcal{P}_3\Big)+$$
$$+\frac{1}{(\mathcal{P}')^5}\Big(8(\mathcal{P}'')^2\mathcal{P}'_{(1}\mathcal{P}_2\mathcal{P}_{3)}+7\mathcal{P}'''\mathcal{P}''\mathcal{P}_1\mathcal{P}_2\mathcal{P}_3\Big)-$$
$$\left.-\frac{8(\mathcal{P}'')^3\mathcal{P}_1\mathcal{P}_2\mathcal{P}_3}{(\mathcal{P}')^6}\right), \qquad (98)$$

where parentheses denote symmetrization (for instance $\mathcal{P}''_{(12}\mathcal{P}_{3)}=\mathcal{P}''_{12}\mathcal{P}_3+\mathcal{P}''_{23}\mathcal{P}_1+\mathcal{P}''_{31}\mathcal{P}_2$).

In KdV critical point i.e. $t_1 = ... = t_{p-1} = 0$, we have $u_c = u_*(t_0, 0, ..., 0) = \sqrt{-t_0}$, and for different derivatives of polynomial $\mathcal{P}(u)$ from (95) one can get

$$\begin{aligned}&\mathcal{P}'(u_c)=2u_c^p,\quad \mathcal{P}''(u_c)=2(2p-1)u_c^{p-1},\\&\mathcal{P}'''(u_c)=6(p-1)^2u_c^{p-2},\quad \mathcal{P}_{k_i}(u_c)=u_c^{p-k_i-1},\\&\mathcal{P}'_{k_i}(u_c)=(p-k_i-1)u_c^{p-k_i-2},\\&\mathcal{P}''_{k_i}(u_c)=(p-k_i-1)(p-k_i-2)u_c^{p-k_i-3},\\&\mathcal{P}_{k_ik_j}(u_c)=0.\end{aligned} \qquad (99)$$

Thus after substitution the expressions (99) in formulas (96), (97) and (98) we obtain

$$\langle O_k\rangle_1 = \frac{p+k}{24}u_c^{-k-2},$$

$$\langle O_{k_1}O_{k_2}\rangle_1 =$$
$$=\frac{(p+2+k_1+k_2)(k_1+k_2)+2p-k_1k_2}{48}u_c^{-k_1-k_2-4},$$
$$\langle O_{k_1}O_{k_2}O_{k_3}\rangle_1=\frac{1}{96}\left(\frac{2k^3}{3}+\frac{k_i^3}{3}+(p+4)k^2+2k_i^2+\right.$$
$$\left.+(6p+8)k+8p-2k_1k_2k_3\right)u_c^{-k_1-k_2-k_3-6}, \quad (100)$$

where $k = k_1 + k_2 + k_3$, $k_i^2 = k_1^2 + k_2^2 + k_3^2$, and $k_i^3 = k_1^3 + k_2^3 + k_3^3$.

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