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Penetration of hot electrons through a cold disordered wire

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We study a penetration of an electron with high energy $E \gg T$ through strongly disordered wire of length $L \gg a$ (a being the localization length). Such an electron can loose, but not gain the energy, when hopping from one localized state to another. We have found a distribution function for the transmission coefficient $\mathcal{T}$. The typical $\mathcal{T}$ remains exponentially small in L/a, but with the decrement, reduced compared to the case of direct elastic tunnelling: $\overline{\ln \mathcal{T}} \approx 0.237 \cdot 2L/a$. The distribution function has a strong tail in the domain of anomalously high $\mathcal{T}$; the average $\overline{\mathcal{T}} \propto (a/L)^2$ is controlled by rare configurations of disorder, corresponding to this tail.

The electronic transport in disordered one-dimensional systems was extensively studied in the past 50 years [1, 2]. It was established that in a non-interacting system all states are localized at arbitrary degree of disorder [3], so that the transmission coefficient $\mathcal{T}$ of a finite system exponentially decays with L and the conductivity σ of a noninteracting system is zero in thermodynamic limit. $\mathcal{T}$ varies from sample to sample, since it depends on random energies and positions of the localized states. For strongly disordered chains the distribution of $s \equiv -(1/\alpha)\ln \mathcal{T}$ is a narrow gaussian:

$$F_0(s) \propto \exp\{-(s-1)^2/(\Delta s)^2\}, \quad \Delta s = B\alpha^{-1/2}, \quad (1)$$

$$\alpha = 2L/a \gg 1, \quad (2)$$

being the principal large parameter of the theory (the measure of the localization strength), $B \sim 1$ being the model-dependent factor [4].

Interactions with a thermostat (e.g., with phonons) lead to a finite equilibrium conductivity σ of the hopping type (see [5]). At some $L \sim L_0(T)$ the exponential L-dependence of the conductance G is changed to $G = \sigma/L$, with σ, exponentially dependent on the temperature T of the system. The specifics of strongly disordered one-dimensional systems was properly taken into account in [6, 7]; it was shown that the conductivity in the variable range hopping regime is controlled by rare fluctuations of the density of states at the fermi level – the "breaks"; as a result

$$\sigma \approx \exp\{-T_0/2T\}, \quad T_0 = 1/ga, \quad (3)$$

where g is the average density of states at the fermi level. Note, that the result (3) does not obey the Mott law $\sigma \propto \exp\{-c(T_0/T)^{1/(d+1)}\}$, valid in dimensions $d \geq 2$ (see [5, 8]), where electrons can easily circumvent the breaks. The nonequilibrium transport in a strong field $\mathcal{E}$ was studied in [9 – 12]. Here the current $I \propto \exp\{-8T_0/e\mathcal{E}a\}$ strongly depends on $\mathcal{E}$, not on T. It is insensitive to the breaks, but, on the other hand, the occupation numbers of localized states are far from equilibrium.

In the present paper we study a different situation, where the current through the system arises due to a *small group* of strongly nonequilibrium high-energy particles, so that the occupation numbers of most electronic states remain essentially in equilibrium. In this letter we do not take into account correlation effects (like Coulomb gap) that arise due to electron-electron interactions.

We consider a disordered wire of length L in equilibrium with two ideal metallic leads (reservoirs) at temperature T and chemical potential ϵ_F (see Fig.1). In

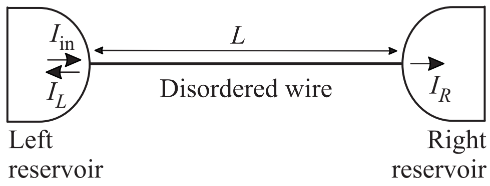

Fig.1. The setup. A disordered wire of length L is in a good contact with two reservoirs, the entire system is equilibrium. A small flux $I_{\rm in}$ of electrons with energy $E \gg T$ is sent to the left end of the wire through the left reservoir, and the part $I_R = \mathcal{T} I_{\rm in}$ finally reaches the right reservoir

the left reservoir, however, a small amount of nonequilibrium particles with energies $E \gg T$ (E is measured with respect to ϵ_F) is injected, so that the current $I_{\rm in}$ reaches the left end of the wire. Possible experimental setups for the realization of the injection process will be discussed elsewhere [13].

Hot electrons, injected into the wire, weakly interact with the thermal bath at temperature T (the bath can be realized, e.g., as phonons or as electronic excitations in the reservoirs) so that their energy is not conserved. However, as long as $E \gg T$, only the processes in which the energy is transferred from electron to the bath, not vice versa, are allowed. The latter statement is only true for not very long wires $L < L_{\rm Mott}(T)$, where $L_{\rm Mott}(T) \sim a(T_0/T)^{1/2}$ is the length of the typical Mott hop at given T. In other words, the temperature should be low enough:

$$T < T_0(a/L)^2. \qquad (4)$$

Under this condition the equilibration does not have chance to occur before the electrons escape from the wire.

Obviously, only the localized states with the energies ε_i in the interval between the Fermi energy and the initial energy E of the injected electron are relevant for our problem. We enumerate them according to their energies: $0 < \varepsilon_1 < \varepsilon_2 < \ldots < \varepsilon_n < E$. These "quasiresonant" levels play an important role in the transport physics, as the electron, travelling through the wire with initial energy E, can make intermediate stops only at these sites. Indeed, all the states with $\epsilon_i < 0$ are occupied, while the states with $\epsilon_i > E$ cannot be reached, as no energy can be absorbed. The spatial positions of quasiresonances are $X_i = Lx_i$, the independent random variables x_i are homogeneously distributed in an interval $0 < x_i < 1$. The number n of quasiresonances is itself a random variable, described by the poissonian distribution $p(n,N) = N^n \exp(-N)/n!$, where $N = LEg$ is the average number of quasiresonances. Thus, each wire is characterized by "the configuration" $\mathcal{C} \equiv \{n, \{x_1, \ldots x_n\}\}$ and the average of any $\mathcal{C}$-dependent quantity $A(\mathcal{C})$ over the ensemble of wires is

$$\overline{A} = \int A(\mathcal{C})\mathcal{D}\mathcal{C} \equiv \sum_{n=0}^{\infty} p(n,N) \prod_{i=1}^{n} \int_0^1 dx_i A(\mathcal{C}). \qquad (5)$$

In particular, the distribution function $F_N(s) = \overline{\delta(s - s(\mathcal{C}))}$. In this letter we focus on the most interesting case $N \gg 1$, when the poissonian distribution is sharp and one can simply average over x_i at fixed $n \approx N$.

How can an electron get from left reservoir to the right one? Besides the obvious possibility of the direct elastic tunnelling (Fig.2a), there are also numerous "inelastic sraircases" (Fig.2b,c), in which an electron makes intermediate stops at certain quasiresonant states, while the excess energy at each hop is transferred to the thermostat. Each staircase $\mathcal{S}$ is characterized by the choice

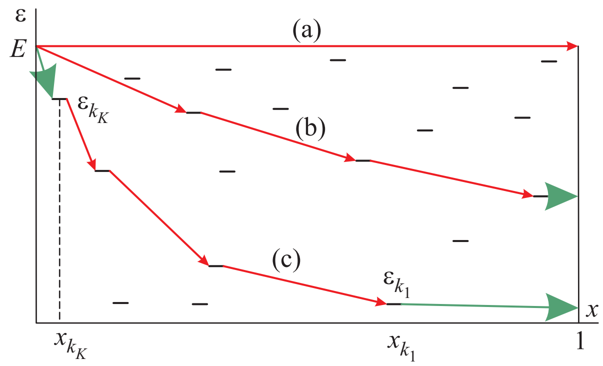

Fig.2. (color online) Possible scenarios for electron passing from the left to the right reservoir. "Natural hops" (occurring with the probability close to unity) are shown by thick green arrows, the non-natural hops (that happen with exponentially small probability) are shown by thin red arrows. (a): Direct elastic tunnelling; (b,c): Inelastic staircases with intermediate stops at quasiresonances

of a subset of K $(0 \le K \le n)$ quasiresonances, with $\varepsilon_{k_1} < \varepsilon_{k_2} < \cdots < \varepsilon_{k_K}$ and $x_{k_1} > x_{k_2} > \cdots > x_{k_K}$.

Each staircase contributes to the transmission:

$$\mathcal{T}(\mathcal{C}) = \sum_{\mathcal{S}} \mathcal{T}(\mathcal{S}|\mathcal{C}), \qquad \mathcal{T}(\mathcal{S}|\mathcal{C}) \propto \exp\{-\alpha s(\mathcal{S}|\mathcal{C})\}, \quad (6)$$

where the summation runs over all the staircases, possible for given configuration $\mathcal{C}$. Under the condition (2) the sum in (6) is dominated by only one – the optimal – staircase $\mathcal{S}_{\rm opt}(\mathcal{C})$ that corresponds to minimal $s(\mathcal{S}|\mathcal{C})$, so that $s(\mathcal{C}) \approx s[\mathcal{S}_{\rm opt}(\mathcal{C})] = \min_{\mathcal{S}} s(\mathcal{S}|\mathcal{C})$. In a typical situation the longest hop in the optimal staircase is the last one, then goes the last but one, etc. Therefore the value of $s(\mathcal{S}|\mathcal{C})$ is controlled by few last hops in $\mathcal{S}$, while the multitude of short hops in the upper part of the staircase are of only secondary importance.

The most natural assumption about the structure of the optimal staircase would be *the scaling hypothesis*: the distribution function $\mathcal{P}_k(\ell_k)$ for random variables $\ell_k = x_{i_k}/x_{i_{k-1}}$ does not depend on k. Such a simple self-similar structure was, however, not observed in our numerical experiments: there is a manifest k-dependence of $\mathcal{P}_k(\ell)$ (see Fig.3).

The statistics of the inelastic staircases is closely related to the well-known mathematical Ulam's problem [14], concerning the statistics of monotonic subsequences a random sequence. However, the analogy is unfortunately, incomplete, so that we did not manage to get profit from this relation.

The explicit expression for $s(\mathcal{S}|\mathcal{C})$ can be found with the help of the stationary master equation for the quasiresonant levels populations f_i:

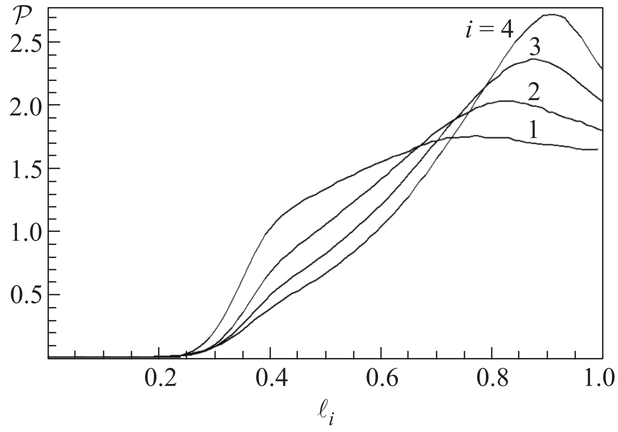

Fig.3. The failure of the scaling assumption: Numerically found distribution functions $\mathcal{P}_i(\ell_i)$ do depend on i

$$P_{L\to i}I_{\rm in} + \sum_{j>i} f_j P_{j\to i} - f_i P_{i\to\rm out} = 0, \qquad (7)$$

$$P_{i\to\rm out} = P_{i\to L} + P_{i\to R} + \sum_{j<i} P_{i\to j}. \qquad (8)$$

The first term in (7) is the incoming flux of particles from the left reservoir; the second term describes the particles, coming to the level i from all other levels with higher energies (hence the condition $j > i$); finally, the third term takes into account all possible escapes from the level i. The system's transmittance is

$$\mathcal{T} \equiv I_R/I_{\rm in} = P_{L\to R} + \sum_i P_{i\to R}(f_i/I_{\rm in}). \qquad (9)$$

The rate of transitions between $j \to i$ is

$$P_{j\to i} = P^{(0)}_{j\to i}(\varepsilon_j - \varepsilon_i)\exp\{-\alpha|x_i - x_j|\}. \qquad (10)$$

Matrix elements of the electron-thermostat interaction, entering $P^{(0)}_{j\to i}$, are smooth power-law functions of the energy transfer $\varepsilon_j - \varepsilon_i$, so that they do not affect the main exponential dependence of $\mathcal{T}$. It means, that if we are interested only in the exponential dependencies, we do not have to take these matrix elements into account, and thus we even do not have to specify any properties of the electron-thermostat interaction! Moreover, the actual values ε_i of the quasiresonances do not enter the exponential factor; the only relevant thing is the hierarchy and spatial positions of the levels. Thus, in the exponential approximation we can write

$$P_{i\to j} \propto \theta(i-j)e^{-\alpha|x_i - x_j|}, \quad P_{i\to\rm out} \propto e^{-\alpha\chi_i}, \qquad (11)$$

$$\chi_i(\mathcal{C}) \equiv \min\left\{x_i, 1 - x_i, \min_{j<i}\{|x_i - x_j|\}\right\}, \qquad (12)$$

being the distance from the i-th quasiresonance to its "natural descendant" – a closest neighbor with lower energy, or to one of the two reservoirs. The solution of the system of equations (7) can be written in a recurrent form:

$$f_i = (P_{L\to i}/P_{i\to\rm out})I_{\rm in} + \sum_{j>i} f_j(P_{j\to i}/P_{i\to\rm out}), \qquad (13)$$

which allows for finding f_i provided all f_j with $j > i$ are already found. According to the exponential approximation, justified by the large parameter (2), any sum, occuring in (9) or in (13), is dominated by a single term with the smallest negative exponent. As a result, the normalized probability for the electron to make a hop $i \to j$ is

$$p(i\to j) = P_{i\to j}/P_{i\to\rm out} = \exp\{-\alpha\Delta s(i\to j)\}, \qquad (14)$$

$$\Delta s(i\to j) = |x_i - x_j| - \chi_i. \qquad (15)$$

Having in mind that $\mathcal{T}(\mathcal{S}|\mathcal{C}) = \prod_{k=1}^{K+1} p(i_k \to i_{k-1})$, we arrive at the result (6) with

$$s(\mathcal{S}|\mathcal{C}) = \sum_{k=1}^{K+1} \Delta s(i_k \to i_{k-1}), \qquad (16)$$

so that $\Delta s(i_k \to i_{k-1})$ has the meaning of the contribution of the k-th hop to the exponent s. In (16) $x_0 = 1$ and $x_{K+1} = 0$ are the positions of the right and the left reservoirs, correspondingly; $\chi_{K+1} = 0$. From (16) it immediately follows that $s(\mathcal{S}|\mathcal{C}) = 1 - \sum_{k=1}^{K+1}\chi_{i_k}$. In principle, our problem can be solved by means of enumeration of all staircases, possible for given configuration, finding the corresponding partial tranmission coefficients and choosing the optimal staircase. The direct solution of the master equations, however, leads to the same result.

Note, that $\Delta s(i_k \to i_{k-1}) = 0$ if the i_{k-1}-th quasiresonance is the natural descendant of the i_k-th one. The corresponding hops we will call "natural hops" in what follows. Clearly, it would be nice to find a staircase, where all the hops $i_k \to i_{k-1}$ (or at least as many of them, as possible) are natural. We will see, however, that such a "natural staircase" is possible to find only for some rare "fortunate configurations"; for typical configurations the natural staircase can not be constructed.

The above observation allows one to find the high transparency (i.e, small s) asymptotics of the distribution function $F_{\overline{N}}(s)$. Suppose that certain configuration $\mathcal{C}$ generates an optimal staircase, in which a sufficiently long sequence of last m hops $k_m \to k_{m-1} \to \cdots \to$ $\to k_1 \to R$ is "natural" (see Fig.4). It means that, if an electron somehow manages to get to the upper level k_m in this sequence, then it makes its descending way to the right reservoir through the rest of the sequence

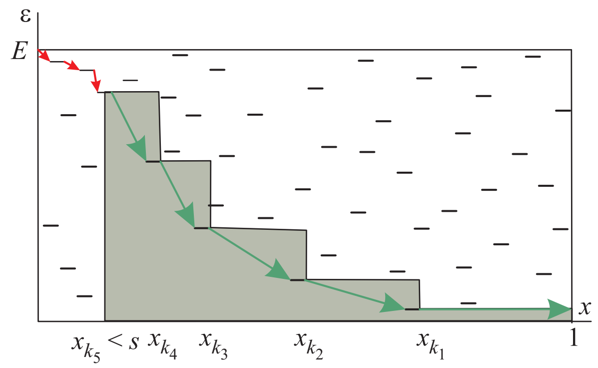

Fig.4. (color online) A "fortunate" configuration (with $m = 5$). The last m "natural" hops in an optimal staircase are shown by thick green arrows. For all $p \le m$ the condition $x_{k_{p-1}}/2 < x_{k_p} < x_{k_{p-1}}$ is fulfilled. Besides that, there are no quasiresonances in the shaded domain

with a probability that is close to unity. Then, if this upper level is close enough to the left reservoir (namely, if $x_{k_m} < s$), then one can guarantee that $s(\mathcal{C}) < s$, no mater whether the preceding short hops in the staircase are natural or not. It seems plausible that just these fortunate configurations with essentially natural optimal staircases constitute the majority of configurations with small $s(\mathcal{C})$.

Let us formulate the conditions necessary for a given configuration $\mathcal{C}$ to be a fortunate one: The last jump (to R) should start from the lowest level (i.e., $k_1 = 1$), and this level should be localized in the right half of the wire: $x_1 > 1/2$. The next to last jump starts from the level, which is the lowest among all quasiresonances i with $0 < x_i < x_1$. This level should be localized in the right half of the interval $(0, x_{k_1})$. In general, the level k_p in the optimal staircase is the lowest one among all the quasiresonances with $0 < x_i < x_{k_{p-1}}$; this level should be in the right half of the interval $(0, x_{k_{p-1}})$. Finally, this should be true for all jumps with $p < m$, where m is determined by the condition $x_{k_m} < s$.

Obviously, the probability for all the m last jumps in the optimal staircase to be natural is 2^{-m}. The only question is, however: what is m? Indeed, m is not determined solely by s: for fixed s m still fluctuates from configuration to configuration.

Let us introduce $\xi_p = 2[x_{k_p} - x_{k_{p-1}}]/x_{k_{p-1}}$. All ξ_p are statistically independent and are homogeneously distributed in the interval $(0,1)$. Then we can write

$$s = \prod_{p=1}^{m}(1+\xi_p)/2, \quad \ln(1/s) = m\ln 2 - \phi. \qquad (17)$$

The distribution $P_m(\phi)$ of the random variable $\phi = \sum_{p=0}^{m}\ln(1+\xi_p)$ is gaussian for $m \gg 1$ and $|\phi - \overline{\phi}| \ll \overline{\phi}$. We will see, however, that the principal contribution to $F_\infty(s)$ at small s comes from the non-gaussian far tail of this distribution. Therefore we will need more accurate expression for $P_m(\phi)$ which is valid also for large deviations. Using the Fourier transformation, we get

$$P_m(\phi) = \int\frac{dq}{2\pi}e^{-iq\phi}\left[\frac{2^{iq+1}-1}{iq+1}\right]^m. \qquad (18)$$

Now we can evaluate

$$F_\infty(s) \approx \sum_m 2^{-m}\int d\phi P_m(\phi)\delta(s - \exp\{\phi - m\ln 2\}) =$$
$$= \sum_m \frac{P_m(m\ln 2 - \ln(1/s))}{s2^m} = \int\frac{dq}{2\pi s}\frac{(1+iq)e^{iq\ln(1/s)}}{2^{-(1+iq)}+iq}.$$

The latter integral is dominated by the residue of that pole of the integrand, which lies in the upper half-plane of q closest to the real axis. This pole is at $q = 2i$, so that we finally arrive at $F_\infty(s) \propto s^{-1}e^{-2\ln(1/s)} \propto s$ and find the asymptotics

$$F_\infty(s) \approx bs, \qquad \text{for } s \ll 1. \qquad (19)$$

where $b \sim 1$ is a universal constant. It should be noted that the dominant contribution comes not from the typical fortunate configurations (those with $\phi \approx \overline{\phi} \equiv \equiv m(2\ln 2 - 1)$), but from the rare "stretched fortunate configurations" with somewhat longer hops, than typical ones. They correspond to $\phi \approx \phi_{\text{opt}} = (m\ln 2)/2 < < \overline{\phi}$. We also stress that, since the optimal deviation $|\phi_{\text{opt}} - \overline{\phi}| \sim \overline{\phi}$ is large, its probability *can not* be described by the Gaussian distribution, which is applicable for the wide range of moderate fluctuations of the variable ϕ: the optimal fluctuation lies in the non-gaussian far tail of the distribution. Disregard of this fact would lead to an incorrect power exponent in (19).

Although the asymptotics (19) describes only a small fraction of the configuration, it turns out to be sufficient for finding the average transmission coefficient:

$$\overline{\mathcal{T}} \approx \int_0^1 bs\exp\{-\alpha s\}ds = b\alpha^{-2} \propto L^{-2}. \qquad (20)$$

The principal contribution to the integral in (20) comes from $s \sim \alpha^{-1} \ll 1$. Thus, the average $\overline{\mathcal{T}}$ is controlled by rare fortunate configurations and is therefore not representative.

The hopping motion of hot particles, accompanied by emission of energy, was first studied in [15] in the context of recombination of photo-excited electron-hole pairs. In the physical situation of [15] it was sufficient to take into account *only the natural hops* since a particle was assumed to be created in the bulk, far from

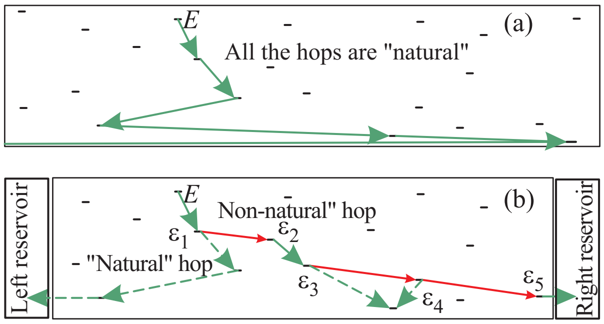

Fig.5. (color online) (a): The “natural paths” consisting of hops to nearest neighbors with lower energy. The particle is initially placed approximately in the middle of the wire, so that in one half of the ensemble of samples the natural path leads to the left reservoir, and it leads to the right reservoir in the other half. (b): The particle is initially placed near the left reservoir, and the natural path leads there in almost all samples. In these (typical) samples, to get to the right reservoir, a particle has to make non-natural hops

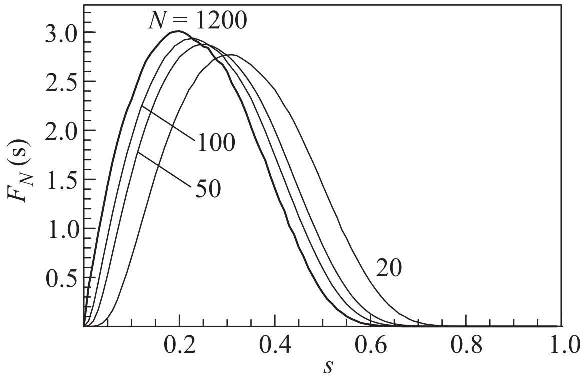

Fig.6. Distribution functions of the logarithm of transparency for different values of average number of quasiresonances $N \equiv ELg$. For $N \gtrsim 200$ the curves practically merge and coincide with $F_\infty(s)$, shown by thick line

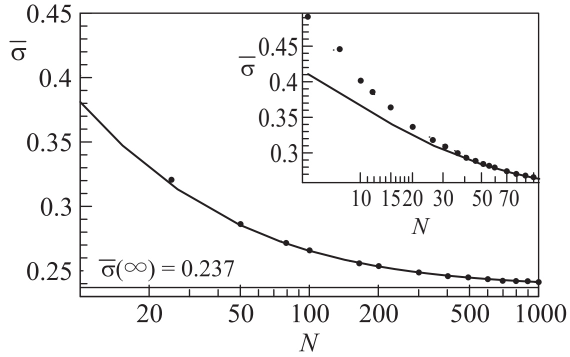

Fig.7. The dependence of the average logarithm of transmittance on N. The asymptotic formula (21) is shown by solid line

the ends of the sample, so that it could not escape to a nearby lead (see Fig.5a). Under these conditions the evolution of particle density profile, *averaged over the the ensemble of samples* could be described by a peculiar diffusion, in which the distribution function of the hops lengths is rescaled by a fixed parameter $q > 1$ after each hop. If one would extend the same approach to the setup where the particle is initially placed near the left absorbing lead, then one easily finds for the probability to reach the right lead (and thus the average transmission coefficient) $\overline{\mathcal{T}} \propto L^{-\beta}$, where the exponent β (as well as the parameter q) depends only on the dimensionality of the space. Note that this result is in agreement with (20). However, as we have already mentioned, $\overline{\mathcal{T}}$ is controlled by very rare anomalous samples with high $\mathcal{T}$, while in a typical sample the natural path would lead to the nearby left lead (see Fig.5b). Therefore, to find the probability to reach the right lead in a typical sample one should necessarily take into account the non-natural hops, that were ignored in [15]; as a result, the typical $\mathcal{T}$ is exponentially small in L. The “diffusional approach”, introduced in [15], being an adequate instrument for finding $\overline{\mathcal{T}}$, is thus useless for the determination of the distribution function of $\mathcal{T}$.

The problem of finding $F_N(s)$ for large N and general s can only be solved numerically. We generated an ensemble of $\sim 10^6$ random configurations $\mathcal{C}$ and calculated corresponding $s(\mathcal{C})$ with the help of the recurrent formula (13). The results of our Monte-Carlo simulations are summarized in Figs.6,7. The distribution functions $F_N(s)$ are wide: the dispersion of s is of order of $\overline{s}$ for all N. For $N \gtrsim 300$ the N-dependence practically disappears: $F_N(s) \approx F_\infty(s)$. The numerically found $F_\infty(s)$ indeed shows a linear low-s asymptotics (19) with $b \approx 29$. This observation supports our assumption about the relevance of the fortunate configurations for the structure of high transparency samples. At $1-s \ll 1$ the distribution $F_N(s)$ vanishes rapidly, our estimates [13] give $F_\infty(s) \propto (1-s)^{\mathrm{const}/(1-s)^2}$, but the accuracy of our simulations is not sufficient to reliably compare this analytic result with numerics.

The average value $\overline{s}(N)$ monotically decreases with N, tending to a finite limit $\overline{s}(\infty) \approx 0.237$ at $N \to \infty$. The convergence is, however, extremely slow: for $N = 1000$ the relative difference is still of order of 2%. The convergence can be improved dramatically if one introduces an asympotic correction according to the empiric formula

$$\overline{s}(N) \approx 0.237 + 0.598 \ln N/N, \qquad \text{for } N \gg 1. \qquad (21)$$

The deviation of experimental $\overline{s}(N)$ from the asymptotics (21) is less than 2% already for $N \sim 30$. The N-dependent corrections are controlled by many short hops in the beginning of the optimal staircase. The scaling hyposesis, mentioned above, would lead to $\overline{s}(N) - \overline{s}(\infty) \propto \propto 1/N$, which is inconsistent with our numerical data. This is another argument against the scaling. In this connection it was interesting to study the N-dependence of an average number $\overline{K}$ of hops in the optimal staircase. The fit of our numerical data (Fig.8) gives

$$\overline{K}(N) \approx 0.39(\ln N)^2 + 2.4, \qquad \text{for } N \gg 1 \qquad (22)$$

which is perfectly consistent with the result (21) and inconsistent with the scaling hypothesis, that would give $\overline{K}(N) \propto \ln N$.

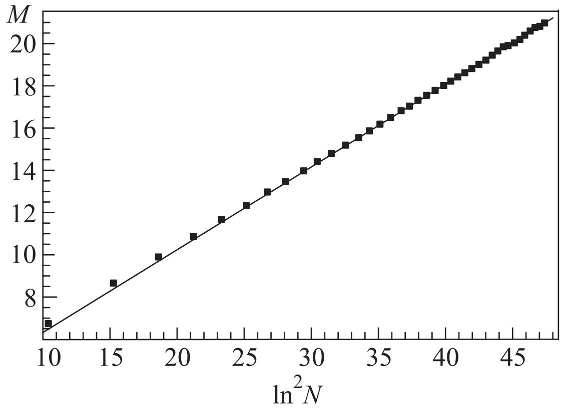

Fig.8. A number of hops $\overline{K}$ in the optimal staircase as a function of N (averaged over $\sim 5 \cdot 10^4$ random configurations)

The dependencies (21) and (22) seem to be closely related to each other. We have invested a lot of efforts in rational explanation of these empirical laws but, unfortunately, so far all our attempts failed. Thus, the elucidation of the structure of the initial part of the optimal staircase and the origin of the large-N corrections still remains a challenge.

In conclusion, we have found the distribution function of the transmission coefficient for the inelastic penetration of a cold disordered wire by a hot electron. The applications of these results to the specific physical effects: the dependence of the current through the system on the chemical potential and/or temperature of the injector, on the position of the chemical potential in the wire, as well as the case of many wires, connected in parellel, will be presented in a long paper to follow [13].

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