Pis'ma v ZhETF, vol. 93, iss. 2, pp. 51 – 55

A remark on the three approaches to 2D Quantum gravity

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Submitted 6 December 2010

The one-matrix model is considered. The generating function of the correlation numbers is defined in such a way that this function coincides with the generating function of the Liouville gravity. Using the Kontsevich theorem we explain that this generating function is an analytic continuation of the generating function of the Topological gravity. We check the topological recursion relations for the correlation functions in the p-critical Matrix model.

1. Introduction. There exist at least three approaches to the 2D Quantum gravity namely the Liouville gravity (LG), the Matrix models (MM) and the Topological gravity (TG). Details and references can be found in reviews e.g. [1, 2].

In this paper we consider the particular p-critical one-matrix model. The correlation functions are defined as derivatives of the free energy function $F(t_0, t_1, \ldots, t_{p-1})$ at some point. The key property of the free energy function of the matrix model is the fulfilment of the string equation and the KdV equations. We consider the expansion of the free energy function at the particular point $t_0 = \mu$, $t_1 = t_2 = \ldots = t_{p-1} = 0$ and choose the particular solution of the string equation. We explain this choice of the boundary conditions in Section 2. This choice is determined by the agreement with the $(2, 2p+1)$ Minimal Liouville gravity.

For the simplest case $p = 2$ the coincidence between the correlation functions in both approaches is straightforward. In the general case the coincidence can be reached after a substitution of variables suggested in [3]. Note, that the coincidence was checked for many cases [3 – 5] but wasn't proved rigorously, due to the fact that correlation functions in the Liouville theory were found only in genus 0 up to four point functions [6].

The term Matrix model referred to related but different things. For example Gross and Migdal in the classical paper [7] compute the correlation functions for the p-critical matrix model as derivatives at different point $t_0 = t_1 = \ldots = t_{p-2} = 0$, $t_{p-1} = t$. Another possibility is to tend (formally) the number of the parameters to infinity (the potential of the model became not a polynomial but a power series). The free energy function depending in infinitely many variables $F(t_0, t_1, \ldots)$ can be defined as the solution of the string and KdV equations. The Witten's conjecture [8] states that this function $F(t_0, t_1, \ldots)$ coincides with the Topological gravity generating function. This conjecture was proved by Kontsevich [9].

It is natural to ask how to compare these two solutions of the string equation namely the generating function of the Topological gravity $F^{\mathrm{TG}}(t_0, t_1, \ldots)$ and the matrix model free energy $F^{\mathrm{MM}}(t_0, t_1, \ldots, t_{p-1})$ mentioned in the second paragraph. It is explained in Section 3 that after the naive vanishing of the extra parameters these functions do not coincide. However, these functions are connected by an nontrivial analytic continuation.

The generating function of the Topological gravity $F(t_0, t_1, \ldots)$ satisfies some partial differential equations which are equivalent to the Topological recursion relations (TRR) for intersection numbers on the moduli spaces of Riemann surfaces. These differential equations involve only finitely many variables. Hence, from the analytic continuation property mentioned above follows that TRR hold for the Matrix model function $F^{\mathrm{MM}}(t_0, t_1, \ldots, t_{p-1})$. We checked these relations in genus 2 (namely the Getzler relations [10]) by a direct computation. The fulfilment of TRR in this Matrix model was checked for the genus 0 in [11], for the genus 1 in [4].

Another recent approach to the relation between the p-critical Matrix models and Topological gravity is given in [12, 13].

2. Preliminaries. *2.1. Liouville gravity.* In this subsection we briefly recall the definition of the correlation functions in the Minimal Liouville gravity. Details can be found in [14, 3].

In this paper we need only the $(2, 2p+1)$ Minimal Liouville Gravity. The total action of the Liouville gravity reads

$$S = S_{\mathrm{L}} + S_{\mathrm{Ghost}} + S_{\mathrm{MM}},$$

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where $S_{\rm MM}$ stands for the $(2,2p+1)$ Minimal CFT action, $S_{\rm Ghost}$ ia a standard ghost action and the Liouville action reads

$$S_L[\phi]=\frac{1}{4\pi}\int\sqrt{\hat{g}}\left[\hat{g}^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi+Q\,\hat{R}\,\varphi+4\pi\mu\,e^{2b\varphi}\right]d^2x,$$

where $b=\sqrt{2/(2p+1)}$ and the parameter μ is interpreted as the cosmological constant. The observables are defined as

$$O_k=\int\Phi_{1,k+1}(x)\,V_{1,-k-1}(x)\,d^2x,$$

where $\Phi_{1,\,k+1},\ V_{1,\,-k-1}(x)$ are certain primary fields of the matter CFT and Liouville theory respectively, $0\le k\le p-1$. The correlation functions defined by the formula

$$\langle\,O_{k_1}...O_{k_N}\,\rangle=\int O_{k_1}...O_{k_N}\,e^{-S[g,\phi]}\,D[g,\phi].$$

It is convenient to define the Liouville gravity generating (or partition) function

$$F^{\rm LG}(\{\lambda\})=\sum_{k_1,k_2,...}\langle O_{k_1}...O_{k_N}\rangle\,\frac{\lambda_{k_1}\dots\lambda_{k_n}}{|{\rm Aut}(k_1,\dots,k_n)|}==\int D[g,\phi]\,e^{-S_\lambda[g,\phi]},\qquad(1)$$

$$S_\lambda[g,\phi]=S[g,\phi]+\sum_{k=1}^{p-1}\lambda_k\,O_k.$$

Consider the genus expansion of the function

$$F^{\rm LG}=F_0^{\rm LG}+F_1^{\rm LG}+\dots,$$

where $F_g^{\rm LG}$ is a generating function of the genus g correlation functions. The functions $F_g^{\rm LG}$ have the scaling property e.g.

$$F_0^{\rm LG}(\rho^{\delta_0}\mu,\rho^{\delta_k}\lambda_k)=\rho^{\frac{2p+3}{2}}F_0^{\rm LG}(\mu,\lambda_k).$$

The gravitation scaling dimensions of the variables

$$\delta_k=\frac{k+2}{2}.\qquad(2)$$

Hence the function $F_0^{\rm LG}$ should has the form

$$F_0^{\rm LG}=\mu^{\frac{2p+3}{2}}g\left(\frac{t_1}{\mu^{3/2}},\frac{t_2}{\mu^2},\dots,\frac{t_{p-1}}{\mu^{(p+1)/2}}\right).\qquad(3)$$

It follows from the definition of the function $F^{\rm LG}$ (1) that the function g is defined by the expansion into the power series.

2.2. Matrix models. In this section we give some basic notion on the Matrix models. Details can be found in the reviews [1, 2] or in the recent paper [4]. The free energy of the one-matrix approach is defined by the matrix integral

$$F(v_k,N)=\log\int dMe^{-{\rm tr}V(M)},$$

where M is a Hermitian $N\times N$ matrix and $V(M)==N\sum v_kM^{2k}$ is a polynomial potential. It is known [1, 2] that the function F can be expanded into the series

$$F(v_k,N)=\sum_{g=0}^{\infty}N^{2-2g}F_g(v_k).\qquad(4)$$

Each term F_g is equal to the sum of contributions of connected surfaces of genus g made of polygons. The one-matrix model possesses a set of multi-critical points, labeled by integer $p=1,2,3,...$ in the space of the "potentials" $V(M)=N\sum v_kM^{2k}$. The p-critical point exists if the number of variables is greater then p i.e. the degree of V is greater than $2p$. We consider the p-critical point for the potential $V(M)=N\sum_{k=1}^{p+1}v_kM^{2k}$. The leading singular term of the function F_g has the scaling property

$$F_g^{\rm sing}[\lambda^{\frac{k+2}{2}}w_k,N]=(\lambda^{p+3/2})^{1-g}F_g^{\rm sing}[w_k,N],\qquad(5)$$

where w_k are certain coordinates centred at the p-critical point. Double scaling limit corresponds to $N\to\infty$ and $w_k\to0$ as $w_k=(N^2\varepsilon^2)^{-\frac{k+2}{2p+3}}t_k$. Usually ε is set to 1, but we keep ε as a parameter in order to consider the genus expansion. In the double scaling limit near the p-critical point the expression (4) looks like

$$F^{\rm sing}[w_k]=\sum_{g=0}^{\infty}N^{2-2g}F_g^{\rm sing}\left((N^2\varepsilon^2)^{-\frac{k+2}{2p+3}}t_k\right)==\varepsilon^{-2}\sum_{g=0}^{\infty}\varepsilon^{2g}F_g^{\rm sing}[t_k],\qquad(6)$$

where we used the scaling property (5) in the second equality. Below we will denote $F_g^{\rm sing}$ as F_g and perform the substitution $\varepsilon^{-2}F\mapsto F$ for simplicity.

2.3. The String equation. The key property of the function F is the fulfilment of the string equation and the KdV equations. Denote by $t_0,t_1,\dots,t_{p-1}$ the KdV coordinates near the p-critical point. All requirements for the KdV coordinates are stated below (for the definition see e.g. [4, Sec. 4] or the reviews [1, 2]). Let

$$u(t_0,t_1,\dots,t_{p-1},\varepsilon)=\partial^2F(t_0,t_1,\dots,t_{p-1},\varepsilon)/\partial t_{p-1}^2.$$

Below we use the notation $t_{-2}=1$, $t_{-1}=0$, $x=t_{p-1}$, $d=\partial/\partial x$. The string equation reads

$$[\hat{P},\hat{Q}]=\varepsilon,\qquad(7)$$

where $\hat{Q} = \varepsilon^2 d^2 + u$ and $\hat{P} = -\sum_{k=1}^{p+1} t_{p-1-k} \hat{Q}_+^{k-1/2}$ are two differential operators. $\hat{Q}_+^{k-1/2}$ is the non-negative part of the pseudo-differential operator $\hat{Q}^{k-1/2}$. The function $u(t_0, t_1, \dots, t_{p-1}, \varepsilon)$ is a solution of (7).

It is known (see e.g. [2, App. A]) that

$$[\hat{Q}_+^{k-1/2}, \hat{Q}] = \varepsilon \frac{dR_k}{dx}, \tag{8}$$

where $R_k(u, u_x, u_{xx}, \dots)$ are the Gelfand–Dikii polynomials in u and its x derivatives. These polynomials are determined by the recursion relation

$$\frac{dR_{k+1}}{dx} = u\frac{dR_k}{dx} + \frac{1}{2}u_x R_k + \frac{\varepsilon^2}{4}\frac{d^3 R_k}{dx^3}, \tag{9}$$

with the boundary conditions $R_1 = u$ and R_k vanish at $u = 0$. The first polynomials have the form

$$R_1 = u, \quad R_2 = \frac{3}{4}u^2 + \varepsilon^2 \frac{1}{4}u_{xx},$$

$$R_3 = \frac{5}{8}u^3 + \varepsilon^2\left(\frac{5}{8}uu_{xx} + \frac{5}{16}u_x^2\right) + \varepsilon^4 \frac{1}{16}u_{xxxx},$$

$$R_k = \frac{(2k-1)!!}{2^k k!}u^k + o(\varepsilon).$$

It follows from (7) and (8) that

$$\sum_{k=1}^{p+1} t_{p-1-k} R_k(u) = -x. \tag{10}$$

We are looking for the solution u in the form

$$u(t, \varepsilon) = u_0 + u_1\varepsilon^2 + u_2\varepsilon^4 + \dots$$

By taking the zeroth order of (10) and rescaling the parameter $t_{p-1-k} \mapsto \frac{2^{k-1}k!}{(2k-1)!!}t_{p-1-k}$ we get

$$\mathcal{P}(u_0) = \sum_{k=0}^{p+1} t_{p-k-1}u_0^k = u_0^{p+1} + \sum_{k=0}^{p-1} t_{p-k-1}u_0^k = 0. \tag{11}$$

One can consider the next orders of ε expansion. It is easy to see that for any g the u_g can be expressed as a rational function in u^*, it's x derivatives and $\mathcal{P}^{(k)}(u^*)$.

It remains to choose the solution of the equation (11). These functions in the variables t_k have the scaling property with the dimensions of the variables

$$\dim(t_k) = \frac{k+2}{2}.$$

Comparing with (2) we conclude that t_0 has the same scale dimension as the cosmological constant μ in the Liouville theory. These parameters have lowest dimension hence its identification is unique up to scalar multiply. We identify t_0 and $-\mu$. Using (3) we get

$$u_0(\mu, t_1, t_2, \dots, t_{p-1}) = \frac{\partial^2 F(t_0, t_1, \dots, t_{p-1})}{\partial t_{p-1}} =$$
$$= \mu^{1/2} g\left(\frac{t_1}{\mu^{3/2}}, \frac{t_2}{\mu^2}, \dots, \frac{t_{p-1}}{\mu^{(p+1)/2}}\right), \tag{12}$$

where the function g is specified by the expansion into the power series. This choice of the root was mentioned in the Introduction.

3. Comparison with Topological gravity. In [15] Witten introduced the Topological gravity. The correlation functions in this theory are defined in terms of intersection numbers on the moduli spaces of complex curves with marked points.

Let $\mathcal{M}_{g,n}$ be the moduli space of complex curves of the genus g with n ordered marked points and $\overline{\mathcal{M}}_{g,n}$ be its Deligne-Mumford compactification. $\overline{\mathcal{M}}_{g,n}$ is the moduli space of stable curves[2)]. $\overline{\mathcal{M}}_{g,n}$ is not a manifold but an orbifold (i.e. locally the quotient of a manifold by a finite group). Its complex dimension is $3g - 3 + n$.

There are natural cohomology classes on such moduli spaces. Let Σ be a stable curve with marked points $x_1, \dots, x_n$. By the definition of Deligne-Mumford compactification the curve Σ may has singularities (double points), but the marked points must be smooth. Thus the cotangent space $T^*_{x_k}\Sigma$ is well defined and the holomorphic bundles $\mathcal{L}_k$ on $\overline{\mathcal{M}}_{g,n}$ with the fiber $T^*_{x_k}\Sigma$ over $(\Sigma, x_1, \dots, x_n)$ can be defined as well. Denote by ψ_k the first Chern class of $\mathcal{L}_k$, $\psi_k = c_1(\mathcal{L}_k)$.

The correlation numbers in the Topological gravity have the form

$$\langle \tau_{d_1} \cdots \tau_{d_n} \rangle = \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \cdots \psi_n^{d_n}, \tag{13}$$

where the genus g is uniquely determined by the condition

$$d_1 + \dots + d_n = \dim \overline{\mathcal{M}}_{g,n} = 3g - 3 + n.$$

The generating function of these correlation numbers reads[3)]

$$\tilde{F}(\tilde{t}_0, \tilde{t}_1, \dots) = \sum_{d_1, d_2, \dots} \langle \tau_{d_1} \cdots \tau_{d_n} \rangle \frac{\tilde{t}_{d_1} \dots \tilde{t}_{d_n}}{|\mathrm{Aut}(d_1, \dots, d_n)|}. \tag{14}$$

[2)]Recall that stable curve means connected, projective curve with no singularities other than double points and with a finite automorphism group. The precise definition and details can be found e.g. in [16]

[3)]We use the notation $\tilde{F}, \tilde{t}_k, \tilde{u}, \tilde{R}_k$ instead of standard F, t_k, u, R_k in order to distinguish the related but not coincident the Topological gravity and the Matrix model objects.

Witten conjectured in [15] the equivalence between two approaches to 2D Quantum gravity namely the Topological gravity and the Matrix models. The precise formulation of this conjecture (see e.g. [8]) states that $\tilde{F}(\tilde{t}_0, \tilde{t}_1, \dots)$ satisfies the KdV hierarchy

$$\frac{\partial \tilde{u}}{\partial \tilde{t}_k} = \frac{\partial \tilde{R}_{k+1}[\tilde{u}]}{\partial \tilde{t}_0}$$

and the string equation

$$\frac{\partial \tilde{F}}{\partial \tilde{t}_0} = \frac{\tilde{t}_0^2}{2} + \sum_{k=0}^{\infty} \tilde{t}_{k+1} \frac{\partial \tilde{F}}{\partial \tilde{t}_k}.$$

Here $\tilde{u} = \partial^2 \tilde{F} / \partial \tilde{t}_0^2$ and each polynomials $\tilde{R}_k$ will coincide with $R_k/2(2k-1)!!$ if we set $u = 2\tilde{u}$ and $\varepsilon = 1$. These polynomials $\tilde{R}_k$ have the form

$$\tilde{R}_1 = \tilde{u}, \qquad \tilde{R}_2 = \frac{\tilde{u}^2}{2} + \frac{\tilde{u}_{xx}}{12},$$

$$\tilde{R}_3 = \frac{\tilde{u}^3}{6} + \frac{\tilde{u}\tilde{u}_{xx}}{12} + \frac{\tilde{u}_x^2}{24} + \frac{\tilde{u}_{xxxx}}{240}, \ \dots \ \tilde{R}_k = \frac{\tilde{u}^k}{k!} + \dots$$

This conjecture was proved by Kontsevich [9]. The string and the KdV equations appear in the Matrix models as an equations for the free energy function.

It is natural to ask how to relate the generating function $\tilde{F}(\tilde{t}_0, \dots)$ in infinitely many variables to the free energy $F(t_0, \dots, t_{p-1})$ of the p-critical Matrix model. For example, assume that we know all intersection numbers (13), how to find the power series expansion of the function g in (12)?

Consider the simplest example $p = 2$. Using the KdV equations, we get from the string equation

$$\frac{\partial \tilde{u}}{\partial \tilde{t}_0} = 1 + \sum_{k=0}^{\infty} \tilde{t}_{k+1} \frac{\partial \tilde{R}_{k+1}[\tilde{u}]}{\partial \tilde{t}_0}.$$

Integrating and restricting it to the genus 0 part, we obtain

$$\sum_{k=0}^{\infty} \tilde{t}_k \frac{\tilde{u}_0^k}{k!} - \tilde{u}_0 = 0.$$

Comparing this with the genus 0 string equation in the Matrix models (11) in the $p = 2$ case we relate the variables

$$\begin{gathered} \tilde{t}_0 = t_1 = x, \ \tilde{t}_1 - 1 = t_0 = -\mu, \\ \tilde{t}_3 = 3!, \ \tilde{t}_2 = \tilde{t}_4 = \tilde{t}_5 = \dots = 0. \end{gathered} \tag{15}$$

Substituting ([16, Prop 4.6.10])

$$\langle \tau_{d_1} \cdots \tau_{d_n} \rangle_0 = \frac{(n-3)!}{d_1! \cdots d_n!}$$

in (14), we obtain the genus 0 generating function

$$\tilde{F}_0(\tilde{t}_0, \tilde{t}_1, \dots) =$$

$$= \sum_{\substack{k_1, k_2, \dots \\ k_0 = k_2 + 2k_3 + 3k_4 + \dots + 3}} \frac{(k_1 + 2k_2 + 3k_3 + \dots)!}{0!^{k_0} 1!^{k_1} \dots} \cdot \frac{\tilde{t}_0^{k_0} \tilde{t}_1^{k_1} \dots}{k_0! k_1! \dots}.$$

With the relation (15), we get

$$\tilde{u}_0(\tilde{t}_0, \tilde{t}_1) = \sum_{\substack{k_1, k_3 \\ k_0 = 2k_3 + 3}} \frac{(k_1 + 3k_3)!}{0!^{k_0} 1!^{k_1} (3!)^{k_3}} \cdot \frac{\tilde{t}_0^{2k_3+1} \tilde{t}_1^{k_1} (3!)^{k_3}}{(2k_3+1)! k_1! k_3!} =$$

$$= \sum_{k_1, k_3} \frac{(k_1 + 3k_3)!}{(2k_3+1)! k_1! k_3!} \tilde{t}_0^{2k_3+1} \tilde{t}_1^{k_1}.$$

Finally, summing on k_1, we obtain

$$\begin{aligned} \tilde{u}_0 &= \sum_{k_3} \frac{(3k_3)!}{(2k_3+1)! k_3!} \frac{\tilde{t}_0^{2k_3+1}}{(1 - \tilde{t}_1)^{3k_3+1}} = \\ &= \frac{x}{\mu} + \frac{x^3}{\mu^4} + 3\frac{x^5}{\mu^7} + \cdots \end{aligned} \tag{16}$$

This function differs from the Matrix models u in formula (12)

$$u_0 = \mu^{1/2} g\left(\frac{x}{\mu^{3/2}}\right) = \mu^{1/2} - \frac{x}{2\mu} - \frac{3x^2}{8\mu^{5/2}} + \cdots \tag{17}$$

Let us consider the expansions (16) and (17) of the roots of the string equation $u^3 - \mu u + x = 0$ at the point $x = 0$. At this point (16) and (17) equal to 0 and $\mu^{1/2}$ respectively. A root of an algebraic equation is locally given by an analytical function in the coefficients of the equation. If the coefficients run round the discriminant set[4)] the roots of equation permute. The group of such permutations is known as the monodromy group. In this case this group is a Galois group of general polynomial and equals to the group of all permutations. Thus (16) and (17) do not coincide but are connected by a nontrivial analytic continuation in x.

It was mentioned in Section 2 that u_g for $g > 0$ can be expressed as rational functions in $u^* = u_0$ and its x derivatives. Therefore, not only $\tilde{u}_0$ transfers to u_0 by an analytic continuation but the whole $\tilde{u} = \sum \varepsilon^{2k} \tilde{u}_k$ transfers to $u = \sum \varepsilon^{2k} u_k$.

For $p > 2$ the argument is quite similar. In the general case u and $\tilde{u}$ are expansions of roots of the string equation $\mathcal{P}(u_0) = 0$ at different points unlike the $p = 2$ case. Still they are connected by an analytic continuation in the variables $t_0, t_1, \dots$

[4)] The set where the discriminant of the corresponding polynomial vanishes.

We are grateful to O. Bershtein, B. Feigin, S. Lando, M. Lashkevich, A. Marshakov, A. Mironov and A. Morozov for useful discussions. We are also grateful to L. Chekhov, B. Eynard and I. Kostov for careful reading the first version of the article and useful remarks.

This research was carried out in the framework of the Federal Program "Scientific and Scientific-Pedagogical personnel of innovational Russia" No 1339, RFBR initiative interdisciplinary project # 09-02-12446-ofi-m and RFBR-CNRS project # PICS-09-02-91064.

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