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# Scalar tilt from broken conformal invariance

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Within recently proposed scenario which explains flatness of the spectrum of scalar cosmological perturbations by a combination of conformal and global symmetries, we discuss the effect of weak breaking of conformal invariance. We find that the scalar power spectrum obtains a small tilt which depends on both the strength of conformal symmetry breaking and the law of evolution of the scale factor.

**1. Introduction and summary.** Primordial scalar perturbations in the Universe are Gaussian (or nearly Gaussian) and have nearly flat power spectrum [1]. These properties are nicely obtained in inflationary theory [2]. Inflation is not unique in this respect, however [3 – 9]. In particular, nearly Gaussian scalar perturbations with flat power spectrum may be generated [7] in a theory of conformal scalar field with negative quartic potential possessing some global symmetry (e.g., $U(1)$). If conformal invariance is exact, the latter mechanism is insensitive to the regime of the cosmological evolution. There is a qualification, though, that there should exist long enough epoch preceding the hot Big Bang stage (so that the horizon problem is solved, at least formally); the mechanism is assumed to operate at that epoch. Once conformal invariance is slightly broken (which may be quite natural), one expects that the spectrum is slightly tilted and depends on the evolution of the scale factor. The purpose of this paper is to demonstrate the latter properties explicitly.

We introduce our model and discuss in general terms the mechanism of the generation of scalar perturbations in Section 2. We then specify the evolution of the scale factor and find the homogeneous background solution describing the scalar field rolling down its potential. The power spectrum is calculated in Section 3. We indeed find that the spectrum is tilted, and that the tilt depends on both the strength of the violation of conformal invariance and the evolution of the scale factor at the epoch when the scalar field rolls down.

**2. Model and background solution.** The model we consider in this paper involves a complex scalar field $\phi$ conformally coupled to gravity. Its dynamics at the epoch of interest is governed by the following action:

$$S[\phi] = \int d^4x\sqrt{-g}\left[g^{\mu\nu}\partial_\mu\phi^*\partial_\nu\phi + \frac{R}{6}\phi^*\phi - V(\phi)\right]. \quad (1)$$

It possesses global $U(1)$ symmetry, and we assume approximate conformal symmetry. To this end, we choose the following scalar potential,

$$V(\phi) = -h^2|\phi|^{4+\alpha}, \quad (2)$$

where $h$ is the coupling constant and the small parameter $\alpha$ characterizes the deviation from conformal invariance. The scalar potential is negative, so the homogeneous background field rolls it down. Without loss of generality, the background field $\phi_c(t)$ can be chosen real.

The mechanism of the generation of scalar perturbations is as follows [7]. One assumes that the Universe is homogeneous and isotropic at the rolling stage and the cosmological evolution is dominated by some other matter during that stage and somewhat later. For $\alpha = 0$, the dynamics of the field $\chi = a\cdot\phi = \chi_1 + i\chi_2$ is independent of the evolution of the scale factor. In that case, the perturbations of the imaginary part $\chi_2$ behave at early times as free massless scalar field in Minkowski space. Later on, when the background $\chi_c$ becomes large, the perturbations of the phase $\delta\theta = \delta\chi_2/\chi_c$ freeze out. For small $h$ and $\alpha = 0$ these perturbations are nearly Gaussian and have flat power spectrum.

If the potential $V(\phi)$ actually has a minimum at some large field value, and the field $\phi$ interacts with matter, the rolling stage terminates at some late time, and the modulus $\phi_1$ relaxes to the minimum. Provided that the energy density of the field $\phi$ is still negligible at that time, the perturbations of the modulus $\delta\phi_1$ are irrelevant for small $h$; what remains are perturbations of the phase. The phase is no longer conformal scalar field; instead, it minimally couples to gravity, as any other Nambu–Goldstone field [10]. The simplest possibility is to assume that modes of cosmologically relevant conformal momenta are superhorizon by the time the modulus relaxes, then the perturbations of the phase remain frozen out[1)].

[1)]An implicit assumption here is that there are no superhorizon modes that grow in time. Such modes, if present, would be

The final part of the scenario is reprocessing the perturbations of the phase into adiabatic perturbations. There are at least two mechanisms for that. The phase can actually be a *pseudo*-Nambu–Goldstone field of relatively small mass, and serve as curvaton [11, 12]. An alternative is that the phase field $\theta$ interacts with matter in such a way that the masses and/or widths of some heavy particles depend on $\theta$ (say, $M = M_0 + \epsilon\theta$ and/or $\Gamma = \Gamma_0 + \epsilon\theta$). If the latter particles, when non-relativistic, temporarily dominate the cosmological expansion, inhomogeneities in $\theta$, and hence in $M$ and/or $\Gamma$ induce adiabatic perturbations $\zeta \sim \delta M/M, \delta\Gamma/\Gamma$ [13, 14]. In either case[2)], to the linear order one has

$$\zeta \propto \delta\theta, \tag{3}$$

so the adiabatic perturbations have flat power spectrum in the model with $\alpha = 0$.

In this paper we are interested in the case of slightly broken conformal invariance, i.e., $\alpha \neq 0$, but

$$|\alpha| \ll 1 .$$

In that case, the scale factor does not drop out from the field equation, and we have to specify the cosmological model. We choose for definiteness contracting Universe filled with matter with stiff equation of state $w > 1$, e.g., some version of the ekpyrotic/bouncing Universe scenario [15, 4, 9]. In terms of conformal time, the evolution of the scale factor is

$$a(\eta) = \mathcal{A}(-\eta)^p, \tag{4}$$

where

$$p = \frac{2}{1+3w} , \quad \mathcal{A} = \text{const} .$$

We begin with the homogeneous background solution that rolls down the scalar potential.

It is convenient to perform the following change of variables:

$$\phi = a^{-\frac{6\alpha}{6+\alpha}}\Psi , \qquad d\xi = a^{-\beta}d\eta , \tag{5}$$

where

$$\beta = \frac{2\alpha}{6+\alpha} .$$

dangerous, as they would make the Universe strongly inhomogenious. However, this problem is avoided at the contracting epoch to be discussed below, provided that the dominant matter has stiff equation of state. This is pricisely the reason for our choice of the model for the background.

[2)]Note that both the pseudo-Nambu–Goldstone curvaton mechanism and modulated decay mechanism require explicit breaking of global $U(1)$.

The motivation for this change of variables is to get rid of the first time derivative in the field equation and simultaneously remove the time variable from the scalar potential. The action in new variables is

$$S[\Psi] = \int d\xi d^3x \left\{ |\Psi'|^2 - a^{2\beta}|\partial_i\Psi|^2 + h^2|\Psi|^{4+\alpha} \quad + \right. \\ \left. + \left[\frac{\alpha(6+2\alpha)}{(6+\alpha)^2}\left(\frac{a'}{a}\right)^2 - \frac{\alpha}{6+\alpha}\frac{a''}{a}\right]\cdot|\Psi|^2\right\}, \tag{6}$$

where prime denotes the derivative with respect to $\xi$, and the second term vanishes for the homogeneous background solution. As the field rolls down its potential towards large values, the last term in (6) becomes subdominant compared to the third term. Hence, at late rolling stage the equation for the background reads

$$\Psi_c'' - (2+\alpha/2)h^2 \cdot \Psi_c^{3+\alpha} = 0, \tag{7}$$

where we assume that the background field $\Psi_c$ is real. The late time attractor solution to this equation is

$$\Psi_c(\xi) = [h(1+\alpha/2)(\xi_* - \xi)]^{-\frac{2}{2+\alpha}} , \tag{8}$$

where $\xi_*$ is the intergation constant. We assume in what follows that

$$\xi_* < 0, \tag{9}$$

so that the entire rolling epoch occurs at the contracting stage.

We have checked numerically that homogeneous solutions to the complete field equation derived from the action (6) indeed approach the asymptotics (8) as the field rolls down its potential.

**3. Scalar perturbations.** Our main purpose is to study linear perturbations of the field $\phi$ in the background (4), (8). In line with the above discussion, we concentrate on the perturbations of the imaginary part. In terms of the variables (5), they obey the following equation (in momentum representation)

$$\delta\Psi_2'' + k^2a^{2\beta}\cdot\delta\Psi_2 - (2+\alpha/2)h^2\Psi_c^{2+\alpha}\cdot\delta\Psi_2 = 0, \tag{10}$$

where we again neglected the contribution that comes from the last term in the action (6). Indeed, the latter contribution is of order $\delta\Psi_2\cdot\alpha/\xi^2$. In view of (8), this is small compared to the last term in the left hand side of Eq. (10), which is of order $\delta\Psi_2\cdot 1/(\xi_*-\xi)^2$.

For the same reason, there is the inequality

$$\beta\frac{a'}{a} \ll \frac{\Psi_c'}{\Psi_c}.$$

This inequality means that the function $a^{2\beta}$ slowly varies in time as compared to $\Psi_c^2$. This suggests the following approach for obtaining the solution to Eq. (10). At early times, when

$$k^2 a^\beta \gg h^2 \Psi_c^{2+\alpha}, \qquad (11)$$

one neglects the last term in the left hand side of Eq. (10) and makes use of the WKB approximation. Note that the inequality (11) can be considered as the condition that the mode of conformal momentum $k$ is in sub-"horizon" regime, the effective "horizon" being due to the evolution of the background field $\Psi_c$. At late times, when the inverse inequality holds, one neglects the second term in Eq. (10). To match the two solutions, one solves Eq. (10) at intermediate times around the "horizon" exit, using the approximation of the time-independent scale factor. In fact, the last two steps can be combined: since the term involving the scale factor is irrelevant at late times anyway, the solution to the equation with the time-independent scale factor is valid both at intermediate and late times.

The WKB solution in the sub-"horizon" regime (11) is

$$\delta\Psi_2 = \frac{1}{(2\pi)^{3/2}\sqrt{2ka^\beta}} \cdot \exp\left(ik\int a^\beta d\xi\right) \hat{A}^\dagger_{\mathbf{k}} + \text{h.c.}, \qquad (12)$$

where $\hat{A}^\dagger_{\mathbf{k}}$ and $\hat{A}_{\mathbf{k}}$ are creation and annihilation operators obeying the standard commutational relations. This solution is equivalent to

$$\delta\chi_2 \equiv \delta(a\phi_2) = \frac{1}{(2\pi)^{3/2}\sqrt{2k}} \cdot \mathrm{e}^{ik\eta} \hat{A}^\dagger_{\mathbf{k}} + \text{h.c.} \qquad (13)$$

This is the expected result, since by neglecting the scalar potential in the sub-"horizon" regime, we are dealing with the theory of massless scalar field conformally coupled to gravity, whose modes are given precisely by (13).

At the time around the "horizon" exit and later we neglect the dependence of $a^\beta$ on time and set

$$a^\beta(\xi) = a^\beta(\xi_\times) \equiv a^\beta_\times ,$$

where the "horizon" crossing time $\xi_\times$ is found from the relation

$$k^2 a_\times^{2\beta} = h^2 \Psi_c^{2+\alpha}(\xi_\times) \approx \frac{1}{(\xi_* - \xi_\times)^2}. \qquad (14)$$

Then Eq. (10) becomes

$$\delta\Psi_2'' + k^2 a_\times^{2\beta} \cdot \delta\Psi_2 - \frac{2+\alpha/2}{(1+\alpha/2)^2} \cdot \frac{1}{(\xi_* - \xi)^2}\delta\Psi_2 = 0. \qquad (15)$$

Its solution that matches (12) at $ka^\beta_\times(\xi_* - \xi) \gg 1$ is

$$\delta\Psi_2 = \frac{1}{4\pi}\sqrt{\frac{\xi_* - \xi}{2}} \cdot H_\nu^{(1)}\left[ka_\times^\beta(\xi_* - \xi)\right] \hat{A}^\dagger_{\mathbf{k}} + \text{h.c.}, \qquad (16)$$

where the index of the Hankel function equals

$$\nu = \frac{1}{2}\cdot\frac{6+\alpha}{2+\alpha}.$$

We find from (16) that the late time asymptotics of the perturbations of the phase is given by

$$\delta\theta \equiv \frac{\delta\Psi_2}{\Psi_c} = \frac{1}{4\pi^{3/2}} \frac{h^{\frac{2}{2+\alpha}}}{k^\nu a_\times^{\nu\beta}} \hat{A}^\dagger_{\mathbf{k}} + \text{h.c.}, \qquad (17)$$

were we neglected correction of order $\alpha$ in the numerical pre-factor. As explained in Ref. [7], the fact that the phase freezes out at late times is due to the global $U(1)$ symmetry of the action (1).

To proceed further, we have to find the "horizon" exit time $\xi_\times$. It is determined by the relation (14). We proceed under the assumption that the relevant scales are superhorizon in conventional sense by the end of the rolling stage. This implies, in particular, that

$$k|\eta_*| \ll 1,$$

where $\eta_*$ is the moment of conformal time corresponding to $\xi = \xi_*$. The latter inequality implies $|\xi_\times| \gg |\xi_*|$, so we obtain from (14) that

$$ka_\times^\beta = |\xi_\times|^{-1} = k^{1-p\beta}\mathcal{A}^\beta,$$

where we keep the terms of the first order in $\alpha$ in the exponent. Finally, according to (17), the power spectrum of the perturbations of the phase is

$$\mathcal{P}(k) = \frac{h^2}{(2\pi)^2}(\mathcal{A}h)^{-\alpha} \cdot k^{\alpha(1+p)}.$$

In accord with our expectation, violation of conformal invariance leads to the tilted power spectrum.

As discussed in Section 2, perturbations of the phase are assumed to be reprocessed into adiabatic scalar perturbations at later epoch of the cosmological evolution, see (3). The adiabatic perturbations inherit the tilt of the spectrum, and we obtain finally

$$n_s - 1 = \alpha(1+p).$$

We see that the tilt depends not only on the conformal symmetry breaking parameter $\alpha$ but also on the evolution of the scale factor parametrized by the exponent $p$ in our case.

We conclude by noticing that breaking of coformal invariance may result, instead of the power-law potential (2), in the logarithmic effective potential

$$V(\phi) = -g^2 \ln \frac{\phi}{\mu},$$

where $\mu$ is assumed to be small. This possibility is, in fact, quite natural from the viewpoint of quantum field theory. The logarithmic potential can be obtained by taking the formal limit $\alpha \to 0$, $\alpha h^2 = \text{const}$. The resulting power spectrum also has logarithmic behavior,

$$\mathcal{P}(k) = \frac{g^2}{(2\pi)^2}(1+p) \ln \frac{k}{k_*}.$$

where $k_* \sim \mu g(a_\times k^p)$ is almost independent of $k$, and $k/k_*$ is large for small $\mu$. Hence, logarithmic violation of conformal invariance implies very mild deviation of the scalar power spectrum from the Harrison–Zeldovich form. This possibility is disfavoured by the cosmological data [1], in view of the absence of tensor perturbations typical for our type of models.

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