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Hints on integrability in the Wilsonian/holographic renormalization group

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The Polchinski equations for the Wilsonian renormalization group in the D–dimensional matrix scalar field theory can be written at large N in a Hamiltonian form. The Hamiltonian defines evolution along one extra holographic dimension (energy scale) and can be found exactly for the subsector of $Tr\phi^n$ (for all n) operators. We show that at low energies independently of the dimensionality D the Hamiltonian system in question reduces to the *integrable* effective theory. The obtained Hamiltonian system describes large wavelength KdV type (Burger–Hopf) equation with an external potential and is related to the effective theory obtained by Das and Jevicki for the matrix quantum mechanics.

Introduction. One of the greatest achievements of the modern fundamental physics is the holographic duality between D–dimensional gauge and $(D+1)$–dimensional gravity theories. The seminal example of this duality is the equality between the quantum generating functional of the correlation functions for the flat space gauge theory and the classical wave functional for the AdS gravity theory [1]. The latter is known as AdS/CFT–correspondence.

The extra dimension in the gravity theory has a natural interpretation as the energy scale in the gauge theory [1] (see e.g. [2] for a review). Moreover, $(D+1)$–dimensional gravity equations of motion can be related to the renormalization group (RG) equations on the gauge theory side [3] (see as well [4–11]).

To understand deeper such a holographic duality, we would like to address the following more general question: what kind of the $(D+1)$–dimensional theories govern RG flows of the large N D–dimensional field theories? In [12] matrix scalar field theory was considered. The Polchinski [13] equations for the Wilsonian RG for this theory were formulated in the subsector of the $\mathrm{Tr}[\phi^n]$ (for all n) operators. At large N the Polchinski equations reduce to the Hamiltonian ones. The latter Hamiltonian system is rather artificial and contains non-local terms. However, in this note we show that in the infrared (IR) limit the RG dynamics is governed by the Hamiltonian[2)] [12]:

$$H = \int_{-\pi}^{+\pi} d\sigma \int d^D x\, \Pi^2 J'. \tag{1}$$

Here $J' = dJ/d\sigma$, $J(T,\sigma,x) = \sum_k \sigma^k J_k(T,x)$, $J_k(T,x)$ are sources for the operators $\mathrm{Tr}[\phi^k(x)]$; $\Pi(T,\sigma,x) = \sum_k \sigma^{-(k+1)}\Pi_k(T,x)$, $\Pi_k(T,x)$ are vaguely speaking vacuum expectation values (VEV) of the operators $\mathrm{Tr}[\phi^k(x)]$ (see below). The sources and VEVs are conjugate to each other via the Poisson bracket: $\{\Pi(T,\sigma,x),\ J(T,\sigma',x')\} \propto \delta(\sigma-\sigma')\,\delta(x-x')$. On the D–dimensional matrix scalar field theory side this can be traced from the Legendre (functional Fourier) relation between the effective actions for the sources and VEVs [12]. The role of the time T for the Hamiltonian system in question is played by the energy scale in the scalar field theory. Note that we observe here the appearance of the one extra (on top of the energy scale) dimension σ conjugate to the number k enumerating the operators $\mathrm{Tr}[\phi^k(x)]$.

Usually in the holographic duality the $(D+1)$–dimensional theory contains gravity. The Hamiltonian system (1) does not contain the symmetric tensor particle because among the $\mathrm{Tr}\phi^l(x)$ operators there is no energy–momentum tensor of the matrix field theory. Hence, naively there is no gravity in the above theory. However, as we discuss at the end of the paper the Hamiltonian system in question might be related to the effective string field theory.

In general, however, it is not clear for us so far under what circumstances and/or how the Hamiltonian equations in question are converted into the Hamiltonian constraint equations of the generally covariant theory. It is not clear for us whether the theory which governs the

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2) Below in this paper we correct the important mistake made by two of us in the paper [12].

RG flow of field theory on the full OPE basis should always (for all large N field theories) be generally covariant or not. This remains to be a challenge for the future work. However, we refer to the RG in question as holographic because, unlike the standard definition of the RG, one can find the theory at any energy scale (high or low one) once he specified this theory at some scale. As well we believe that AdS/CFT–correspondence is of the same origin.

The Hamiltonian flow along the original D directions, denoted in (1) as x, is trivial. I.e. basically we deal with continuous collection of non–linear mechanics enumerated by the index x. So one can just forget this index and study the dynamics of the system with the Hamiltonian $H = \int_{-\pi}^{+\pi} d\sigma \Pi^2 J'$.

One of the goals of this paper is to show that the Hamiltonian system in question is integrable. And to show that it is equivalent to the effective field theory derived by Das and Jevicki for the matrix quantum mechanics [14] (see as well [15, 16]).

Holographic formulation of the large N Wilsonian RG. We consider the D–dimensional Euclidian matrix scalar field theory whose action in the Fourier transformed form can be written as:

$$\mathcal{S}[\phi] = -\frac{N}{2} \times$$
$$\times \int \mathrm{Tr}\left[\phi(p)\,(p^2+m^2)\;\; K_\Lambda^{-1}(p^2)\,\phi(-p)\right]\, d^D p +$$
$$+ N\sum_{l=0}^{\infty}\int d^D k_1 \dots d^D k_l\, \mathrm{Tr}\left[\phi(k_1)\;\;\dots\,\phi(k_l)\right] \times$$
$$\times J_l(-k_1-\dots-k_l). \qquad (2)$$

Here J_l are the sources. We assume that there is some momentum cut–off imposed, i.e. $K_\Lambda(p^2)\sim 1$ as $p^2 \ll \Lambda^2$, while $K_\Lambda(p^2)\to 0$ as $p^2 \gg \Lambda^2$. As well we assume that $J_l(k) = 0$ for all l for $|k| > \lambda$, where λ is some low energy scale (where we measure our physics).

The Polchinski equations for the theory in question follow from the RG invariance of the functional integral Z [13]:

$$\Lambda\frac{d\mathcal{S}_I[\phi]}{d\Lambda} = -\frac{1}{2}\int \frac{d^D p}{p^2+m^2}\Lambda\frac{dK_\Lambda(p^2)}{d\Lambda} \times$$
$$\times\left[N^{-1}\frac{\delta^2\mathcal{S}_I[\phi]}{\delta\phi^{ij}(-p)\delta\phi^{ji}(p)} + \frac{\delta\mathcal{S}_I[\phi]}{\delta\phi^{ij}(p)}\frac{\delta\mathcal{S}_I[\phi]}{\delta\phi^{ji}(-p)}\right],$$
$$\mathcal{S}_I = N\sum_{l=0}^{\infty}\int d^D k_1 \dots d^D k_l\, \mathrm{Tr}\left[\phi(k_1)\;\;\dots\,\phi(k_l)\right] \times$$
$$\times J_l(-k_1-\dots-k_l), \qquad (3)$$

i.e. this equation is supposed to specify the scale Λ dependence of the sources J_l to fulfill the equation $dZ/d\Lambda = 0$.

To proceed, we verify the following relations:

$$\mathrm{Tr}\left[\frac{\delta^2\mathcal{S}_I}{\delta\phi(p)\delta\phi(-p)}\right] =$$
$$= \sum_{a=1}^{\infty}\sum_{s=0}^{a-1}\int_{k_{(a-1)}}(a+1)\mathrm{Tr}\left[\phi(k_1)\;\;\dots\phi(k_s)\right]\times$$
$$\times\,\mathrm{Tr}\left[\phi(k_{s+1})\;\;\dots\phi(k_a)\right] J_{a+1}(-k_{(a-1)});$$
$$\mathrm{Tr}\left[\frac{\delta\mathcal{S}_I}{\delta\phi(p)}\frac{\delta\mathcal{S}_I}{\delta\phi(-p)}\right] = \sum_{l,j=1}^{\infty}\int_{q_{(j-1)}\,k_{(l-1)}}(n\cdot l)\times$$
$$\times\,\mathrm{Tr}\left[\phi(q_1)\,\dots\,\phi(q_{j-1})\;\;\phi(k_1)\,\dots\,\phi(k_{l-1})\right]\times$$
$$\times\, J_j(-p-q_1-\dots-q_{j-1})\,J_l(p-k_1-\dots-k_{l-1}); \quad (4)$$

Here we use the same notations as in [12]: $J_l(-k_{(l)}) := J_l(-k_1-\dots-k_l)$ and $\int_{k_{(l)}} := \int d^D k_1 \dots d^D k_l$. The first among these two equations is obtained after the regrouping summands under the sum.

If one substitutes the obtained expressions for the variations of $\mathcal{S}_I$ into the RG equation (3), he has to make two observations. First, the operators containing derivatives of ϕ's do not appear on the RHS of the RG equation. That is true simply because operators containing powers the momenta k and q are not generated from the functional derivatives of $\mathcal{S}_I$ from (3). Second, one encounters higher trace operators on the RHS of the RG equation (3). It seems that to close the obtained system one has to add to (2) sources for higher trace operators as well (and of cause in general one has to add operators containing derivatives). That is the standard approach in the Wilsonian RG: at the end one has to use Operator Product Expansion and the completeness of the basis of operators.

However, in the large N limit one can take a different way of addressing the problem [11]. In this limit one has the factorization property, $\langle\prod_n O_n\rangle = \prod_n\langle O_n\rangle + \mathcal{O}\left(1/N^2\right)$, which is valid for any choice of the bare action used for performing the quantum average $\langle\dots\rangle$. Due to the factorization property at large N one can express any operator of the complete OPE algebra basis as a polynomial in the single trace operators. I.e. in the limit in question single trace operators form a basis through which any operator in the complete OPE algebra can be expressed algebraically. In this note we of cause consider only a subspace of the single trace operators, which contains only the operators without derivatives.

Thus, our idea is that the first step to obtain the closed system of equations for the single trace operators is to take the quantum average of (3) [12]. To do that we separate the field ϕ into two contributions $\phi = \phi_0 + \varphi -$

the low energy one ϕ_0, which solves the equations of motion following from (2)[3], and the high energy ones φ, which contain harmonics between λ and Λ. In the quantum average we take the functional integral over the φ. In [12] we have used the averaging with respect to the Gaussian theory. However, all our formulas remain valid even if we average with the use of an interacting action. The reason for that is the validity of the factorization property at large N.

Thus, substituting $\mathcal{S}_I$ to the equation (3), then taking its quantum average and using the factorization property, one obtains:

$$\sum_{l=1}^{\infty}\int_{k_{(l)}} T_l(\{k_l\})\dot{J}_l(-k_{(l)}) =$$
$$= -\frac{1}{2}\int_p \frac{\dot{K}_\Lambda(p^2)}{p^2+m^2}\Bigg[N^{-1}\sum_{a=1}^{\infty}\sum_{s=0}^{a-1}\int_{k_{(a-1)}}(a+1)\times$$
$$\times T_{a-s-1}(\{k_{a-s-1}\})T_s(\{k_s\})J_{a+1}(-k_{(a-1)}) +$$
$$+\sum_{l,j=1}^{\infty}\int_{q_{(j-1)}k_{(l-1)}}(l\cdot j)T_{l+j-2}(\{k_{l-1}\},\{q_{j-1}\})\times$$
$$\times J_l(-k_{(l-1)}-p)J_j(-q_{(j-1)}+p)\Bigg], \tag{5}$$

where overdot means $\Lambda\, d/d\Lambda$ and $T_a(\{k_a\}) := \langle \mathrm{Tr}[\phi(k_1)\ldots\phi(k_a)]\rangle$.

It is probably worth stressing now that after the quantum averaging the operators containing differentials of ϕ do appear in the RG evolution. However, they contribute to the RG flow of J's under consideration within $T_l = \langle\mathrm{Tr}\phi^l\rangle = \mathrm{Tr}\phi_0^l +$ higher derivative $(\partial\phi_0)$ terms. Note that eq. (5) contains only J's and T's.

The equation (5) still is not closed since the RG dynamics for the sources J depends on the VEVs T. However, one can close the system by deriving the RG equations for the VEVs T as well. To derive the above Polchinski equation, we have used the fact that the effective action $W(J) = \log Z$ is cutoff independent. However, we can make the Legendre transform from $W(J)$ to the effective action $I(T)$. The latter should not depend on the cutoff as well. The Legendre transform is a functional Fourier transform with respect to which sources J and operators T play the role of generalized coordinates and momenta. Hence, from the RG invariance of $I(T)$ we expect to get a conjugate (via Legendre transform) equation describing the RG dynamics of T. That is explicitly checked in perturbation theory in [11].

[3]This field contains harmonics lower than the low energy scale λ, because as we assume the sources $J_l(p)$ are zero when the modulus of their argument is grater than λ.

Now taking equation (5) and the corresponding Legendre conjugate one for T's, then equating to zero each term in front of every operator T_l in (5) and in front of every J_l in the Legendre transformed equation, we obtain the system of equations of the Hamiltonian form $\dot{J}_l(-k_{(l)}) = \delta H(J,T)/\delta T_l(\{k_l\})$ and $\dot{T}_l(\{k_l\}) = -\delta H(J,T)/\delta J_l(-k_{(l)})$, where

$$H = -\frac{1}{2}\int_p \frac{\dot{K}_\Lambda(p^2)}{p^2+m^2}\times$$
$$\times\Bigg[N^{-1}\sum_{a=1}^{\infty}\sum_{s=0}^{a-1}\int_{k_{(a-1)}}(a+1)T_{a-s-1}(\{k_{a-s-1}\})\times$$
$$\times T_s(\{k_s\})J_{a+1}(-k_{(a-1)}) +$$
$$+\sum_{l,j=1}^{\infty}\int_{q_{(j-1)}k_{(l-1)}}(l\cdot j)T_{l+j-2}(\{k_{l-1}\},\{q_{j-1}\})\times$$
$$\times J_l(-k_{(l-1)}-p)J_j(-q_{(j-1)}+p)\Bigg]. \tag{6}$$

Such an approach within perturbation theory was verified in [11]. So far we have been making identity transformations (up to the derivation of the equations for T).

The "Hamiltonian system" under consideration, although being closed, is rather artificial at least because it does not have fixed dimensionality. All the terms in (5) are relevant in the *UV limit* ($\lambda \sim \Lambda$). Which is necessary to restore the proper β–functions of the sources. However, we are going to argue that in the *IR limit* ($\lambda \ll \Lambda$) the second term on the RHS of (5) is irrelevant. To see this note that the cut–off function $K_\Lambda(p^2)$ is taken to be close to the step–function $\theta(p^2-\Lambda^2)$. Hence, its derivative is close to the delta–function $\delta(p^2-\Lambda^2)$. The moduli of the arguments of $T_a(\{k_{(l-1)}\},\{q_{(j-1)}\})$ are all smaller than $\lambda \ll \Lambda$, i.e. $|k_l| < \lambda$ and $|q_j| < \lambda$ in the second line of (6) for all l and j. Hence, the moduli of the arguments of the sources $J_l(k)$ in the second term in RHS of (6) are very close to Λ. But, since $J_l(k) = 0$, when $|k| \geq \lambda$, the last term in (6) is negligible.

A possible brief explanation of what is going on here is as follows[4]. Among the operators $\mathrm{Tr}\phi^l$ the relevant and marginal in the IR are those having $l \leq [l_{cr}]$, where $[l_{cr}]$ is the integer part of $l_{cr} = 2D/(D-2)$. All the rest of the operators are irrelevant in the IR limit. But, in the theory under consideration, even the relevant and marginal operators are suppressed in the IR in comparison with the Gaussian part of the action: even in the UV limit we see that quantum corrections to the conformal dimensions of these operators deviate them towards being less relevant, while marginal ones are converted

[4]It is probably worth stressing here that using RG property one can change $d/d\log\Lambda$ for $d/d\log\lambda$.

into slightly irrelevant. Thus, in the IR limit we are in the vicinity of the Gaussian theory and the RG dynamics is governed by the Hamiltonian (6) without the second term[5)].

Thus, if we neglect the second term in the IR limit the Hamiltonian in question reduces to:

$$H=\int_{q_1 q_2}\sum_{l,s=0}^{\infty}[(l+s+2)\Pi_l(q_1)\Pi_s(q_2)J_{l+s+2}(-q_1-q_2)]. \tag{7}$$

The time in this theory is related to the scale factor as follows: $T=\int_p \frac{K(p^2/\Lambda^2)}{p^2+m^2}$. The canonical momenta $\Pi_n(p)$ in the momentum representation are as follows: $\Pi_n(p)=\frac{1}{N}\int_{p_1\dots p_n}\delta(p-p_1-\dots-p_n)\langle\mathrm{Tr}[\phi(p_1)\dots\phi(p_n)]\rangle$, where the average is taken over the high–energy harmonics φ.

To represent the Hamiltonian (7) in the ultra–local form we introduce the Fourier transform of the J_k and Π_k harmonics: $J(T,\sigma,x)=\sum_k \sigma^k J_k(T,x)$, $\Pi(T,s,x)=\sum_k \sigma^{-(k+1)}\Pi_k(T,x)$. After such a substitution the Hamiltonian acquires the simple form:

$$H=\int_{-\pi}^{\pi}d\sigma\int d^D x\,\Pi^2 J'. \tag{8}$$

The dynamics of this Hamiltonian system along the D directions (denoted by x) is trivial and we can skip the x dependence of J and Π. Then the corresponding equations of motion are: $\dot J = 2\Pi J'$, $\dot\Pi = 2\Pi\Pi'$. The equations of motion for the field $J(t,\sigma)$ only have the form:

$$-\partial_t\left(\frac{\dot J}{J'}\right)+\frac{1}{2}\partial_\sigma\left(\frac{\dot J^2}{J'^2}\right)=0. \tag{9}$$

This equation (for $P=\dot J/J'$) is referred to as the inviscid Burger's or Hopf equation and is know to be integrable.

RG, matrix models, effective theory and strings. In the IR limit (when we can neglect the second term in (6)) in the coordinate representation the Polchinski equation (5) can be written as

$$\int d^D x\sum_k \Pi_k(x)\dot J_k(x)=$$
$$=-\int d^D x\sum_{k,l}(k+l+2)\Pi_k(x)\Pi_l(x)J_{k+l+2}(x), \tag{10}$$

where overdot means the differentiation with respect to the above defined "time" T.

To establish a correspondence of the holographic RG equations and effective field theory of Das and Jevicki we redefine the sources and the conjugated momenta in Polchinski equation according to their natural conformal dimensions: $J_k(x)=\Lambda^{\alpha k}g_k, \Pi_k(x)=\Lambda^{-\alpha k}p_k$, where $\alpha=(2-D)/2$. Then the above Hamiltonian transforms to:

$$H=\left[\sum_{k,l}(k+l+2)\,p_k\,p_l\,g_{k+l+2}+\sum_k kp_k\,g_k\right]. \tag{11}$$

Here we have a different definition of time $t = -\frac{D-2}{2}\log\Lambda$. The case of $D=2$ is special: one should take $\alpha=-1/2$ and $\Pi_k(x)=\Lambda^{-k/2+1}p_k(x)$. The time then becomes $t_{D=2}=-1/2\log\Lambda$ and the Hamiltonian is the same.

Making the Fourier transform as above and introducing the new variable $s=e^{i\sigma}$, we obtain:

$$H=\int ds\left[p^2\,g'+s\,p\,g'\right]. \tag{12}$$

The equation of motion for the field $g(t,s)$ is as follows:

$$-s-\partial_t\left(\frac{\dot g}{g'}\right)+\frac{1}{2}\partial_s\left(\frac{\dot g^2}{g'^2}\right)=0. \tag{13}$$

With the obvious change of variables $P=\dot g/g'$ this equation can be rewritten in a rather simple way:

$$\dot P = P\,\partial_s\,P - s. \tag{14}$$

Such an equation follows from the Hamiltonian system, which in the standard notations looks as

$$H=\frac{1}{4\pi}\int ds\left[\frac{1}{3}P_+^3-\left(s^2-\mu\right)\,P_+\right]\times$$
$$\times\{P_+(s),\,P_+(s')\}\propto\partial_s\,\delta(s-s'), \tag{15}$$

and is considered in [15, 16]. It describes the effective field theory for the matrix quantum mechanics. As well this Hamiltonian describes large wavelength KdV type (Burger–Hopf) equation with an external $(-s^2)$ potential. It is known to be integrable. The relation of the matrix field theory to the string theory in a bit different setting was discussed as well in [17].

Conclusion. We see that the same effective Das–Jevicki field theory appears in the same matrix quantum mechanics (field theory in general) in two seemingly different approaches. The present form of the Das–Jevicki theory follows from the Gaussian matrix quantum mechanics (field theory in general), while in our case we

[5)]The observations of the last two paragraphs correct the mistake which was made by two of us in the earlier paper [12], where we have represented the second term of the above "Hamiltonian system" in the ultra–local form.

obtain it as the IR limit of the RG dynamics, where the matrix field theory under consideration flows to the Gaussian form. Furthermore, the two ways of the derivation of this effective field theory should be related. We as well as Das and Jevicki, in fact, derive the same effective field theory on the same phase space — VEVs of the operators and their sources in the matrix field theory. Note that the extra coordinate s in the Hamiltonian has the same origin as in the paper [14]: there they have been using a bit different basis of operators $\mathrm{Tr}[e^{i\,k\,\phi}]$ from ours, which probably explains the reason why we did not get the Das–Jevicki Hamiltonian directly. The time in the Das–Jevicki theory appears to be one of the x's of the matrix field theory under consideration. However, in the string field theory interpretation of the Das–Jevicki effective theory the time is related to the Liouville mode which defines the scale on the string world–sheet, while in our case it is just the energy scale.

One of the interesting questions is to understand the meaning of the solutions of the full Hamiltonian system in question from the point of view of the RG. Note that Wilsonian RG requires to define the value of the field g (sources) at the initial value of "time" Λ, while the value of the the momentum p should be fixed at the final "time" λ. Note that we observe seemingly unexpected kind of RG dynamics because in the IR limit the RG flow goes in cycles described by the angle–action variables of the integrable system in question. The RG dynamics of such a type was predicted in [18].

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