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# Compact equation for gravity waves on deep water

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Using the fact of vanishing four waves interaction for water gravity waves for 2D potential fluid we were able significantly simplify well-known but cumbersome Zakharov equation. Hamiltonian of the obtained equation is very simple and includes only fourth order nonlinear term. It rises the question about integrability of the free surface hydrodynamics. This new equation is very suitable as for analytic study as for numerical simulation.

**1.Zakharov's Equation.** A one-dimensional potential flow of an ideal incompressible fluid with a free surface in a gravity field fluid is described by the following set of equations:

$$\begin{aligned} \phi_{xx} + \phi_{zz} &= 0 \qquad (\phi_z \to 0, z \to -\infty), \\ \eta_t + \eta_x \phi_x &= \phi_z \Big|_{z=\eta}, \\ \phi_t + \frac{1}{2}(\phi_x^2 + \phi_z^2) + g\eta &= 0 \Big|_{z=\eta}, \end{aligned}$$

here $\eta(x,t)$ – is the shape of a surface, $\phi(x,z,t)$ – is a potential function of the flow and $g$ – is a gravitational constant. As was shown in [4], the variables $\eta(x,t)$ and $\psi(x,t) = \phi(x,z,t)\Big|_{z=\eta}$ are canonically conjugated, and satisfy the equations

$$\frac{\partial \psi}{\partial t} = -\frac{\delta H}{\delta \eta}, \qquad \frac{\partial \eta}{\partial t} = \frac{\delta H}{\delta \psi}.$$

Here $H = K + U$ is the total energy of the fluid with the following kinetic and potential energy terms:

$$K = \frac{1}{2}\int dx \int_{-\infty}^{\eta} v^2 dz, \qquad U = \frac{g}{2}\int \eta^2 dx.$$

It is convenient to introduce normal complex variable $a_k$:

$$\eta_k = \sqrt{\frac{\omega_k}{2g}}(a_k + a^*_{-k}), \qquad \psi_k = -i\sqrt{\frac{g}{2\omega_k}}(a_k - a^*_{-k}),$$

here $\omega_k = \sqrt{gk}$ – is the dispersion law for the gravity waves, and Fourier transformations $\psi(x) \to \psi_k$ and $\eta(x) \to \eta_k$ are defined as follows:

$$f_k = \frac{1}{\sqrt{2\pi}}\int f(x) e^{-ikx} dx, \quad f(x) = \frac{1}{\sqrt{2\pi}}\int f_k e^{+ikx} dk.$$

Hamiltonian can be expanded in an infinite series in powers of $a_k$ (see [4, 5])

$$H = H_2 + H_3 + H_4 + \dots$$

This variable $a_k$ satisfies the equation

$$\frac{\partial a_k}{\partial t} + i\frac{\delta H}{\delta a_k^*} = 0,$$

where

$$\begin{aligned} H_2 &= \int \omega_k a_k a_k^* dk, \\ H_3 &= \int V^k_{k_1 k_2}\{a_k^* a_{k_1} a_{k_2} + a_k a^*_{k_1} a^*_{k_2}\}\delta_{k-k_1-k_2} dk dk_1 dk_2 + \\ &+ \frac{1}{3}\int U_{k k_1 k_2}\{a_k a_{k_1} a_{k_2} + a_k^* a^*_{k_1} a^*_{k_2}\}\delta_{k+k_1+k_2} dk dk_1 dk_2, \end{aligned}$$

$$\begin{aligned} V^k_{k_1 k_2} &= \frac{g^{\frac{1}{4}}}{8\sqrt{\pi}}\left(|k|^{\frac{1}{4}} L_{k_1 k_2} - |k_2|^{\frac{1}{4}} L_{-k k_1} - |k_1|^{\frac{1}{4}} L_{-k k_2}\right), \\ U_{k k_1 k_2} &= \frac{g^{\frac{1}{4}}}{8\sqrt{\pi}}\left(|k|^{\frac{1}{4}} L_{k_1 k_2} + |k_2|^{\frac{1}{4}} L_{k k_1} + |k_1|^{\frac{1}{4}} L_{k k_2}\right), \end{aligned}$$

$$L_{k k_1} = \frac{1}{|k k_1 k_2|^{\frac{1}{4}}}(k k_1 + |k||k_1|).$$

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Fourth order part of Hamiltonian is the following:

$$H_4 = \frac{1}{2}\int W^{k_3k_4}_{k_1k_2} a^*_{k_1} a^*_{k_2} a_{k_3} a_{k_4} \delta_{k_1+k_2-k_3-k_4} dk_1 dk_2 dk_3 dk_4 +$$
$$+\frac{1}{3}\int G^{k_4}_{k_1k_2k_3} (a^*_{k_1} a^*_{k_2} a^*_{k_3} a_{k_4} + \text{c.c.}) \delta_{k_1+k_2+k_3-k_4} dk_1 dk_2 dk_3 dk_4 +$$
$$+\frac{1}{12}\int R_{k_1k_2k_3k_4} (a^*_{k_1} a^*_{k_2} a^*_{k_3} a^*_{k_4} + \text{c.c.}) \delta_{k_1+k_2+k_3+k_4} dk_1 dk_2 dk_3 dk_4 .$$

Here $W^{k_3k_4}_{k_1k_2}$, $G^{k_4}_{k_1k_2k_3}$ and $R_{k_1k_2k_3k_4}$ are equal to:

$$W^{k_3k_4}_{k_1k_2} = -\frac{1}{32\pi}\Big[M^{k_1k_2}_{-k_3-k_4} + M^{-k_3-k_4}_{k_1k_2} - M^{k_1-k_3}_{k_2-k_4} - $$
$$- M^{k_2-k_3}_{k_1-k_4} - M^{k_1-k_4}_{k_2-k_3} - M^{k_2-k_4}_{k_1-k_3}\Big].$$

$$G^{k_4}_{k_1k_2k_3} = -\frac{1}{32\pi}\Big[M^{k_3-k_4}_{k_1k_2} + M^{k_2-k_4}_{k_1k_3} + M^{k_1-k_4}_{k_2k_3} - $$
$$- M^{k_1k_2}_{k_3-k_4} - M^{k_1k_3}_{k_2-k_4} - M^{k_2k_3}_{k_1-k_4}\Big].$$

$$R_{k_1k_2k_3k_4} = -\frac{1}{32\pi}\Big[M^{k_3k_4}_{k_1k_2} + M^{k_2k_4}_{k_1k_3} + M^{k_2k_3}_{k_1k_4} + $$
$$+ M^{k_1k_4}_{k_2k_3} + M^{k_1k_3}_{k_2k_4} + M^{k_1k_2}_{k_3k_4}\Big].$$

Here

$$M^{k_3k_4}_{k_1k_2} = |k_1k_2|^{\frac{3}{4}} |k_3k_4|^{\frac{1}{4}} (|k_1+k_3| + |k_1+k_4| +$$
$$+ |k_2+k_3| + |k_2+k_4| - 2|k_1| - 2|k_2|).$$

Now one can apply canonical transformation from variables $a_k$ to $b_k$ to exclude non resonant cubic terms along with non resonant fourth order terms with coefficients $G^{k_4}_{k_1k_2k_3}$ and $R_{k_1k_2k_3k_4}$. This transformation up to the accuracy $O(b^5)$ has the form [4, 6, 7]:

$$a_k = b_k + \int \Gamma^k_{k_1k_2} b_{k_1} b_{k_2} \delta_{k-k_1-k_2} dk_1 dk_2 -$$
$$- 2\int \Gamma^{k_2}_{kk_1} b^*_{k_1} b_{k_2} \delta_{k+k_1-k_2} dk_1 dk_2 +$$
$$+ \int \Gamma_{kk_1k_2} b^*_{k_1} b^*_{k_2} \delta_{k+k_1+k_2} dk_1 dk_2 +$$
$$+ \int B^{k_2k_3}_{kk_1} b^*_{k_1} b_{k_2} b_{k_3} \delta_{k+k_1-k_2-k_3} dk_1 dk_2 dk_3 +$$
$$+ \int C^{k_3}_{kk_1k_2} b^*_{k_1} b^*_{k_2} b_{k_3} \delta_{k+k_1+k_2-k_3} dk_1 dk_2 dk_3 +$$
$$+ \int S_{kk_1k_2k_3} b^*_{k_1} b^*_{k_2} b^*_{k_3} \delta_{k+k_1+k_2+k_3} dk_1 dk_2 dk_3. \qquad (1)$$

Here

$$B^{k_2k_3}_{kk_1} = \Gamma^{k_1}_{k_1-k_2} \Gamma^{k_3}_{kk_3-k} + \Gamma^{k_1}_{k_3k_1-k_3} \Gamma^{k_2}_{kk_2-k} -$$
$$- \Gamma^{k}_{k_2k-k_2} \Gamma^{k_3}_{k_1k_3-k_1} - \Gamma^{k_1}_{k_3k_1-k_3} \Gamma^{k_2}_{k_1k_2-k_1} -$$
$$- \Gamma^{k+k_1}_{kk_1} \Gamma^{k_2+k_3}_{k_2k_3} + \Gamma_{-k-k_1kk_1} \Gamma_{-k_2-k_3k_2k_3} + \tilde{B}^{kk_1}_{k_2k_3},$$

$$\Gamma^k_{k_1k_2} = -\frac{V^k_{k_1k_2}}{\omega_k - \omega_{k_1} - \omega_{k_2}}, \quad \Gamma_{kk_1k_2} = -\frac{U_{kk_1k_2}}{\omega_k + \omega_{k_1} + \omega_{k_2}},$$

$\Gamma^k_{k_1k_2}$ and $\Gamma_{kk_1k_2}$ provides vanishing of cubic terms in the new Hamiltonian and $\tilde{B}^{kk_1}_{k_2k_3}$ is an arbitrary function satisfying the following symmetry conditions:

$$\tilde{B}^{kk_1}_{k_2k_3} = \tilde{B}^{k_1k}_{k_2k_3} = \tilde{B}^{kk_1}_{k_3k_2} = -(\tilde{B}^{k_2k_3}_{kk_1})^*.$$

$\tilde{B}^{kk_1}_{k_3k_2}$ manages $2 \leftrightarrow 2$ coefficient in the new Hamiltonian. Coefficients $C^{k_3}_{kk_1k_2}$ and $S_{kk_1k_2k_3}$ provide vanishing $3 \leftrightarrow 1$ and $4 \leftrightarrow 0$ terms in the Hamiltonian.

After transformation (1) the Hamiltonian acquires the following form:

$$H = \int \omega_k b_k b^*_k dk + \frac{1}{2}\int [T^{k_2k_3}_{kk_1} - (\omega_k + \omega_{k_1} - \omega_{k_2} - \omega_{k_3}) \times$$
$$\times \tilde{B}^{k_2k_3}_{kk_1}] b^*_k b^*_{k_1} b_{k_2} b_{k_3} \delta_{k+k_1-k_2-k_3} dk dk_1 dk_2 dk_3 + \ldots . \qquad (2)$$

If $\tilde{B}^{k_2k_3}_{kk_1} = 0$, equation (2) is known as Zakharov's equation. Here $T_{kk_1,k_2k_3}$ satisfies the symmetry conditions:

$$T^{k_2k_3}_{kk_1} = T^{k_2k_3}_{k_1k} = T^{k_3k_2}_{kk_1} = T^{kk_1}_{k_2k_3}$$

and has the form:

$$T^{k_2k_3}_{kk_1} = W^{k_2k_3}_{k_1k} -$$
$$- \left[\frac{V_{kk_2k-k_2} V_{k_3k_1k_3-k_1}}{\omega_{k_2} + \omega_{k-k_2} - \omega_k} + \frac{V_{kk_2k-k_2} V_{k_3k_1k_3-k_1}}{\omega_{k_1} + \omega_{k_3-k_1} - \omega_{k_3}}\right] -$$
$$- \left[\frac{V_{k_1k_2k_1-k_2} V_{k_3kk_3-k}}{\omega_{k_2} + \omega_{k_1-k_2} - \omega_{k_1}} + \frac{V_{k_1k_2k_1-k_2} V_{k_3kk_3-k}}{\omega_k + \omega_{k_3-k} - \omega_{k_3}}\right] -$$
$$- \left[\frac{V_{kk_3k-k_3} V_{k_2k_1k_2-k_1}}{\omega_{k_3} + \omega_{k-k_3} - \omega_k} + \frac{V_{kk_3k-k_3} V_{k_2k_1k_2-k_1}}{\omega_{k_1} + \omega_{k_2-k_1} - \omega_{k_2}}\right] -$$
$$- \left[\frac{V_{k_1k_3k_1-k_3} V_{k_2kk_2-k}}{\omega_{k_3} + \omega_{k_1-k_3} - \omega_{k_1}} + \frac{V_{k_1k_3k_1-k_3} V_{k_2kk_2-k}}{\omega_k + \omega_{k_2-k} - \omega_{k_2}}\right] -$$
$$- \left[\frac{V_{k+k_1kk_1} V_{k_2+k_3k_2k_3}}{\omega_{k+k_1} - \omega_k - \omega_{k_1}} + \frac{V_{k+k_1kk_1} V_{k_2+k_3k_2k_3}}{\omega_{k_2+k_3} - \omega_{k_2} - \omega_{k_3}}\right] -$$
$$- \left[\frac{U_{-k-k_1kk_1} U_{-k_2-k_3k_2k_3}}{\omega_{k+k_1} + \omega_k + \omega_{k_1}} + \frac{U_{-k-k_1kk_1} U_{-k_2-k_3k_2k_3}}{\omega_{k_2+k_3} + \omega_{k_2} + \omega_{k_3}}\right]. \qquad (3)$$

At this moment we approach to the key point of this article. Namely, appropriate choice of $\tilde{B}^{kk_1}_{k_2k_3}$ in (2) is based on two things:

1. The coefficient $T^{k_2k_3}_{kk_1}$ is identically equal to zero on the resonant manifold [1]:

$$k + k_1 = k_2 + k_3,$$
$$\omega_k + \omega_{k_1} = \omega_{k_2} + \omega_{k_3}, \qquad (4)$$

with nontrivial solution:

$$
\begin{aligned}
k &= a(1+\zeta)^2, \\
k_1 &= a(1+\zeta)^2\zeta^2, \\
k_2 &= -a\zeta^2, \\
k_3 &= a(1+\zeta+\zeta^2)^2;
\end{aligned} \tag{5}
$$

here $0<\zeta<1$ and $a>0$

2. Also we consider waves moving in the same direction, that allows us to consider only positive wavenumbers $k$. This assumption came from numerical simulations [2, 3].

This fact and that observation allow drastically simplify Hamiltonian. One can vanish cumbersome expression for $T_{kk_1}^{k_2k_3}$ in (2) keeping only its diagonal part. This diagonal part corresponds to trivial four-wave scattering

$$k_2 = k_1, \quad k_3 = k, \quad \text{or} \quad k_2 = k, \quad k_3 = k_1. \tag{6}$$

It is equal to

$$T_{kk_1} = T_{kk_1}^{kk_1} = \frac{1}{4\pi}|k||k_1|\left(|k+k_1| - |k-k_1|\right). \tag{7}$$

Using this diagonal part one can construct the following function (with tilde):

$$
\begin{aligned}
\tilde{T}_{k_2k_3}^{kk_1} = \Bigg[&\frac{1}{2}(T_{kk_2}+T_{kk_3}+T_{k_1k_2}+T_{k_1k_3}) - \\
&- \frac{1}{4}(T_{kk}+T_{k_1k_1}+T_{k_2k_2}+T_{k_3k_3})\Bigg]\theta(kk_1k_2k_3),
\end{aligned} \tag{8}
$$

$$\theta(x) = \begin{cases} 0, & \text{if } x<0; \\ 1, & \text{if } x>0. \end{cases}$$

This function. $\tilde{T}_{k_2k_3}^{kk_1}$, has been intoduced already in our work [8], however only now we discovered possibility of essential simplification of Zakharov equation and its Hamiltonian. Let us consider the case when all waves move in the same direction. It mean that all $k$ have the same sing. Thus, let $k_i>0$, and

$$b \simeq e^{i(kx-\omega t)}.$$

Then $\tilde{T}_{k_2k_3}^{kk_1}$ can be significantly simplified, *modulus for* $|k+k_1|$ *along with* $|k|$ *and* $|k_1|$ *can be dropped.* Now

$$\hat{T}_{kk_1} = \frac{1}{4\pi}kk_1\left(k+k_1-|k-k_1|\right).$$

Simple calculations end up with

$$
\begin{aligned}
\tilde{T}_{k_2k_3}^{kk_1} = &\frac{1}{8\pi}[kk_1(k+k_1)+k_2k_3(k_2+k_3)] - \\
&-\frac{1}{8\pi}(kk_2|k-k_2|+kk_3|k-k_3|+ \\
&+k_1k_2|k_1-k_2|+k_1k_3|k_1-k_3|).
\end{aligned} \tag{9}
$$

**2. Compact equation.** Let us make choice for $\tilde{B}_{k_2k_3}^{kk_1}$ as follows:

$$\tilde{B}_{k_2k_3}^{kk_1} = \frac{T_{k_2k_3}^{kk_1} - \tilde{T}_{k_2k_3}^{kk_1}\theta(kk_1k_2k_3)}{\omega_k+\omega_{k_1}-\omega_{k_2}-\omega_{k_3}}. \tag{10}$$

It makes four-wave coefficient in (2) equal to $\tilde{T}_{k_2k_3}^{kk_1}$. Note that expression (10) for $\tilde{B}_{k_2k_3}^{kk_1}$ has no singularity on resonance manifold. Using following relations for $\hat{K}$ and space derivative

$$kb_k^* \Leftrightarrow i\frac{\partial}{\partial x}b^*(x), \tag{11}$$

$$kb_k \Leftrightarrow -i\frac{\partial}{\partial x}b(x), \tag{12}$$

$$|k-k_2|b_k^*b_{k_2} \Leftrightarrow \hat{K}(|b(x)|^2) \tag{13}$$

hamiltonian can be easily written in $X$-space:

$$
\begin{aligned}
\mathcal{H} = &\int b^*\hat{\omega}b\,dx + \frac{i}{16}\int\left[b^{*2}\frac{\partial}{\partial x}(b'^2) - b^2\frac{\partial}{\partial x}(b^{*\prime 2})\right]dx - \\
&-\frac{1}{4}\int |b|^2\hat{K}(|b'|^2)dx.
\end{aligned} \tag{14}
$$

After integrating by parts Hamiltonian acquires very nice form:

$$\mathcal{H} = \int b^*\hat{\omega}b\,dx + \frac{1}{4}\int |b'|^2\left[\frac{i}{2}(bb'^* - b^*b') - \hat{K}|b|^2\right]dx. \tag{15}$$

This is the main result of the article. Corresponding equation of motion is the following:

$$
\begin{aligned}
i\frac{\partial b}{\partial t} = \hat{\omega}b + &\frac{i}{8}\left[b^*\frac{\partial}{\partial x}\left(b'^2\right) - \frac{\partial}{\partial x}\left(b^{*\prime}\frac{\partial}{\partial x}b^2\right)\right] - \\
&-\frac{1}{4}\left[b\hat{K}(|b'|^2) - \frac{\partial}{\partial x}[b'\hat{K}(|b|^2)]\right].
\end{aligned} \tag{16}
$$

Along with usual quantities such as energy and both momenta equation (16) conserves action or number of waves:

$$N = \int |b|^2 dx.$$

**3. Some solutions.** *3.1. Monochromatic wave.* Monochromatic wave with arbitrary amplitude $B_0$

$$b(x) = B_0e^{i(k_0x-\omega_0t)} \tag{17}$$

is the simplest solution of (16). Indeed, plugging (17) in to the equation (16) one can get the following relation

$$\omega_0 = \omega_{k_0} + \frac{1}{2}k_0^3|B_0|^2. \tag{18}$$

Recalling transformation from $a_k$ to $b_k$ one can see that for waves with small amplitude ($a_k \simeq b_k$)

$$|B_0|^2 = \frac{\omega_{k_0}}{k_0}\eta_0^2,$$

and relation (18) coincides with well known Stokes correction to the frequency due to finite wave amplitude

$$\omega_0 = \omega_{k_0}\left(1 + \frac{1}{2}k_0^2|\eta_0|^2\right). \tag{19}$$

*3.2. Modulational instability of monochromatic wave.* We consider perturbation to the solution

$$b = B_0 e^{i(k_0 x - \omega_0 t)},$$

where

$$B_0 = \frac{1}{\sqrt{2\pi}}\int b_{k_0} e^{i(k_0 x - kx)}dx$$

and

$$\omega_0 = \omega_{k_0} + \frac{1}{2}|B_0|^2 k_0^3, \qquad \frac{1}{4\pi}|b_{k_0}|^2 k_0^3 = \frac{1}{2}T_{k_0 k_0}^{k_0 k_0}|b_{k_0}|^2.$$

Perturbed solution has the following form:

$$b \Rightarrow (b_{k_0} + \delta b_{k_0+k}e^{-i\Omega_k t} + \delta b_{k_0-k}e^{-i\Omega_{-k}t})e^{-i\omega_0 t} \tag{20}$$

with the following condition:

$$\Omega_k = -\Omega_{-k}.$$

Suppose $\delta b_{k_0+k}$ grows as

$$\delta b_{k_0+k} \Rightarrow \delta b_{k_0+k}e^{\gamma_k t}$$

one can easily obtain the following formula for $\gamma_k$:

$$\gamma_k^2 = \left[-d(k) - \frac{3|B_0|^2}{4}k_0 k^2\right]\left[d(k) + |B_0|^2 k_0\left(k_0 - \frac{|k|}{2}\right)^2\right]. \tag{21}$$

If we introduce steepness of the carrier wave $\omega_{k_0}\mu^2 = |B_0|^2 k_0^2$ and approximate $d(k)$ as

$$d(k) \simeq -\frac{1}{8}\omega''_{k_0}k^2 = -\frac{1}{8}\omega_{k_0}\frac{k^2}{k_0^2},$$

then growth rate is equal to:

$$\gamma_k^2 = \frac{1}{8}\frac{\omega_{k_0}^2}{k_0^4}(1 - 6\mu^2)k^2\left[\mu^2\left(k_0 - \frac{|\mathbf{k}|}{2}\right)^2 - \frac{k^2}{8}\right].$$

This expression for growth rate is more accurate than usually derived from nonlinear Schrodinger equation. The difference is seen from two terms marked with boldface.

*3.3. Breathers.* Equation for amplitude of the wave train can be easily derived from (16). Let us introduce envelope $B(x,t)$ so that

$$b(x,t) = B(x,t)e^{i(k_0 x - \omega_0 t)}. \tag{22}$$

$B(x,t)$ is also normal hamiltonian variable and Hamiltonian for it is the following:

$$\begin{aligned}\mathcal{H} = &\int B^*(\hat{\omega}_{k_0+k} - \omega_{k_0})B dx + \\ &+ \frac{1}{4}\int |B' + ik_0 B|^2 \times \\ &\times \left\{\frac{i}{2}(B(B'^* - ik_0 B^*) - B^*(B' + ik_0 B)) - \hat{K}|B|^2\right] dx.\end{aligned} \tag{23}$$

Breather is the solution equation with Hamiltonian (23) in the following form:

$$B(x,t) = B(x - Vt)e^{-i\Omega t}. \tag{24}$$

$V$ is close to linear group velocity.

**4. Conclusion.** Simple equation describing evolution of 1D water waves is derived. Derivation of this equation is based on the important property of vanishing four-wave interaction for gravity water waves. This property allows to simplify drastically well-known Zakharov's equation for water waves, which is very cumbersome. Written in $X$-space instead of $K$-space, it allows further analytical and numerical study. Simple Hamiltonian which is obtained after canonical transformation rises the question about integrability of the equations for potential flow of fluid in the gravity field. It remains still open.

This new equation can be generalized for the "almost" 2D waves, or "almost" 3D fluid. When considering waves slightly inhomogeneous in transverse direction, one can think in the spirit of Kadomtsev–Petviashvili equation for Korteveg–de-Vries equation, namely one can treat now frequency $\omega_k$ as two dimensional, $\omega_{k_x,k_y}$, while leaving coefficient $\tilde{T}_{k_2 k_3}^{k k_1}$ in (9) not dependent on $y$. Now $b$ depends on both $x$ and $y$:

$$\begin{aligned}\mathcal{H} = &\int b^*\hat{\omega}_{k_x,k_y} b dx dy + \\ &+ \frac{1}{4}\int |b'_x|^2\left[\frac{i}{2}(bb'^*_x - b^* b'_x) - \hat{K}_x|b|^2\right] dx dy.\end{aligned} \tag{25}$$

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