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On the corrections to the Casimir effect depending on the resolution of measurement

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The Casimir force $\mathcal{F} = -\frac{\pi^2 \hbar c}{240 a^4}$, which attracts to each other two perfectly conducting parallel plates separated by the distance a in vacuum, is one of the blueprints of the reality of vacuum fluctuations. Following the recent conjecture, that quantum fields should be described in terms of the fields depending on the resolution of measurement, rather than the position alone [1], we derive the correction to the Casimir energy depending on the ratio of the plate displacement amplitude to the distance between plates.

The Casimir force is the result of the difference of the vacuum zero point energy of the two different configurations: the rectangular volume $L_x \times L_y \times a$ bounded by two parallel conducting walls, and that of the same volume not bounded by conducting walls. In the former case, the electromagnetic field bounded between conducting walls is said to be dimensionally quantized, while in the latter case the frequency spectrum is continuous. The energy difference between these two configurations cannot be measured directly, but it varies with the variation of the gap a, and this variation can be measured as the Casimir force.

In 1948 Casimir conjected that the force between two parallel conducting planes depends only on two universal constants, $\hbar$ and c, and the distance between the plates a [2]. The first attempt to measure the Casimir force has been undertaken in 1958 [3]. Later the Casimir force have been measured with an atomic force microscope [4, 5]. A lot of studies related to the Casimir effect are being carried out in different branches of nanomechanics and photonics now, see e.g. [6, 7] and references therein for recent review.

In the dimensionally quantized case the zero point energy of the electromagnetic field between conducting plates is

$$E_Q = \frac{\hbar c}{2} \sum_\alpha |k_\alpha| = \frac{\hbar c}{2} \int \frac{L_x L_y dk_\parallel^2}{(2\pi)^2} \Big[|k_\parallel| + + 2 \sum_{n=1}^\infty \left(k_\parallel^2 + \frac{\pi^2 n^2}{a^2} \right)^{1/2} \Big], \qquad (1)$$

where factor 2 with the sum over discrete spectrum accounts for two possible polarizations of the electromagnetic field; $k_\parallel \equiv (k_x, k_y)$. The energy of the same field free of any boundary conditions is expressed as integral over continuous spectra

$$E_0 = \frac{\hbar c}{2} \int \frac{L_x L_y dk_\parallel^2}{(2\pi)^2} \int_{-\infty}^{\infty} 2 \frac{dk_z}{(2\pi)} \left(k_\parallel^2 + k_z^2 \right)^{1/2}. \qquad (2)$$

Both integrals (1), (2) are evidently infinite, but their difference

$$\mathcal{E} = \frac{E_Q - E_0}{L_x L_y}, \qquad (3)$$

known as the Casimir energy, can be regularized if the r.h.s. of the equations (1), (2) are multiplied by some cutoff function $f(k)$, such that $f(0) = 1$ and $f(k \gg 1/a_0) \to 0$, where $1/a_0$ is the inverse size of atom. This specific choice accounts for the fact, that the walls are metallic plates made of real atoms, rather than of an ideal conductor.

Such choice of the cutoff function f is practically appropriate, but does not relate the calculated ultra-violet infinities to what is really measured at the finite scale of measurement δ, see Fig. 1. The conjecture, relating the quantum fields to the aperture function of the measurement taken with resolution δ was given in [1]. The aperture function $g(x) = -x e^{-x^2/2}$ leads in one-loop approximation, up to appropriate rescaling, to the cutoff function

$$f(k) = e^{-4\delta^2 k^2}. \qquad (4)$$

After the choice (4) the regularized Casimir energy is (see § 3.2.4 of [8]):

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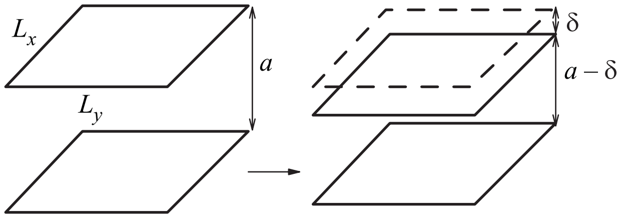

Fig. 1. Shift of the upper conducting wall from $z = a$ to $z = a - \delta$ changes the Casimir energy from $\mathcal{E}(a,\delta)$ to $\mathcal{E}(a-\delta,\delta)$

$$\mathcal{E}=\frac{\hbar c\pi^2}{4a^3}\left[\frac{F(0)}{2}+F(1)+F(2)+\ldots-\int_0^\infty dnF(n)\right],$$
$$F(n)=\int_0^\infty du\sqrt{u+n^2}e^{-4\pi^2(\delta^2/a^2)(u+n^2)}=$$
$$=\frac{\sqrt{\pi}}{2\left(2\pi\delta/a\right)^3}\left\{1-\operatorname{erf}\left(\frac{2\pi\delta}{a}n\right)+\right. \quad (5)$$
$$\left.+4\sqrt{\pi}\exp\left[-\left(\frac{2\pi\delta}{a}\right)^2n^2\right]\frac{\delta}{a}n\right\}.$$

The difference between the sum and the integral in (5) is evaluated by Euler–Maclaurin formula

$$\frac{1}{2}F(0)+F(1)+\ldots-\int_0^\infty dnF(n)=$$
$$=-\frac{1}{2!}B_2F'(0)-\frac{1}{4!}B_4F'''(0)-\ldots,$$

where B_n are the Bernoulli numbers. This gives the corrections to the Casimir energy

$$\mathcal{E}(a,\delta)=-\frac{\hbar c\pi^2}{720a^3}\left[1+\frac{2}{7}\left(\frac{2\pi\delta}{a}\right)^2+\frac{3}{28}\left(\frac{2\pi\delta}{a}\right)^4+\ldots\right], \quad (6)$$

and the Casimir force

$$\mathcal{F}(a,\delta)=-\frac{\hbar c\pi^2}{240a^4}\left[1+\frac{10}{21}\left(\frac{2\pi\delta}{a}\right)^2+\frac{1}{4}\left(\frac{2\pi\delta}{a}\right)^4+\ldots\right], \quad (7)$$

respectively.

We would like to emphasize, that if the conjecture of the previous paper [1] is physically correct, and so the resolution of measurement δ is a real physical parameter, which constraints maximal momenta of the field fluctuations, rather than being a formal regularization parameter, the deviations from the standard results should be observed if we compare two measurements with the same gap between planes, but different resolution.

In Fig. 2 below we present the comparison of the "exact" Casimir force between two plates in vacuum ($\delta = 0$), and that calculated according to (7) with $\delta/a = 0.1$.

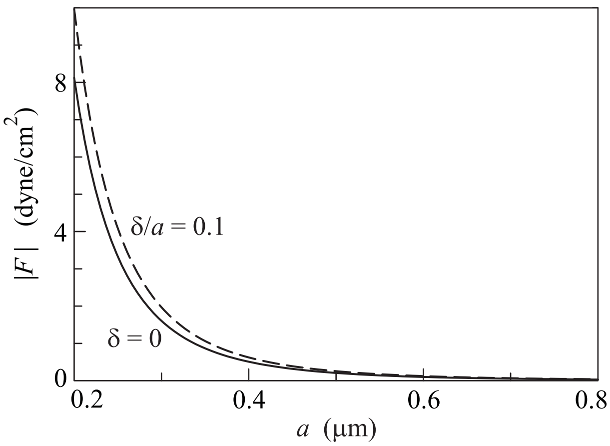

Fig. 2. Deviation of Casimir force between two plates of unit area in vacuum. The solid line corresponds to the "exact" Casimir force ($\delta = 0$), the dashed line corresponds to the scale-dependent Casimir force with $\delta/a = 0.1$

The dependence of the Casimir force on the cutoff parameter have been already suggested in the framework of the quantum field theory problem scaled to a condensed matter effective theory [9], where the inter-atomic distance plays the same role as the Planck length in high energy physics. It was concluded that actual Casimir force should be stronger than that predicted by conventional Casimir theory. The dependence on the cutoff scale also raises some criticism against the application of regularization methods to Casimir effect, specially for spherical geometry [10].

Interestingly, in present experiments the separation between plates is corrected by the factor $1+(\delta/a)^2$, derived from the Taylor expansion of the Casimir force, to account for the r.m.s. fluctuations of the random environment [11]. Our correction to the Casimir force due to the finite resolution of the measurement, given by Eq. (7) is also consistent with the limits posed by the precise measurement of the Casimir force given in [12] with the resolution $\delta/a \approx 4\cdot10^{-3}$. The choice of the scale parameter δ as a displacement amplitude is one of the possible simplifications. (Here we do not regard the dynamical effects [13].) For real experiments an important characteristics of the setup is a ratio of the boundary layer thickness to the distance between plates, which may be of order $h/a \sim 10^{-4}$–10^{-2} [14]. With decreasing of the boundary plate thickness, according to the Lifshitz theory [15], the electron plasma of metal boundaries become utmost transparent for high frequency photons [16], and the ratio h/a plays a role of another cutoff parameter.

An experimental study of the corrections to the Casimir force is certainly a challenging problem, where the dielectric permeability of the media should be taken into account at finite temperatures [5, 17, 18].

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