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Schwinger pair creation in multilayer graphene

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The low energy effective field model for the multilayer graphene (at ABC stacking) in external Electric field is considered. The Schwinger pair creation rate and the vacuum persistence probability are calculated using the semiclassical approach.

1. Introduction. The Schwinger mechanism of electron-positron pair creation in electric field was first investigated in [1]. The rate of pair production depends on electric field and is so small that is not observable for the experimentally allowed values of the electric field. In the condensed matter systems the situation may be different. For example, in graphene [2, 3] the large value of the effective coupling constant opens the possibility for the pair creation process to be observed [4, 5]. In graphene monolayer the pair creation rate may be calculated using the approach of [1]. However, a different approaches were also used (see, for example, [4, 6] and references therein). The approach described in [6] was also applied to the bilayer graphene [7, 8].

In the present paper we calculate the pair production rate in multilayer graphene. We consider the simplest case of ABC-stacking described by the two-band pseudospin Hamiltonian with the chirality index J equal to the number of layers [9]. In our calculations we rely on the method developed in [4, 10] and in [11]. A similar approach was also applied to He-3 [12]. Within this approach, which was originally applied to monolayer graphene, we develop the semiclassical approximation (for the alternative ways to apply semiclassical technique to the fermionic models see [13, 14]). This approximation gives results identical to the results obtained via the exact solution of the Schrodinger equation for the case of monolayer. Our results obtained in the multilayer graphene are checked with the more traditional semiclassical approach described in [6–8].

The paper is organized as follows. In Section 2 we describe the one-particle Schrodinger equation that appears in the given problem. In Section 3 we introduce appropriate boundary conditions. In Section 4 the semiclassical approximation for the one-particle Schrodinger equation is introduced. In Section 5 we check the results obtained in Section 4 via the semiclassical approach described in [6–8]. In Section 6 we compare our results with the exact ones for the case of graphene monolayer. In Section 7 we calculate the vacuum persistence probability and the pair production rate for the field – theoretic model of multilayer graphene. In Section 8 we end with the conclusions.

2. One-particle Schrodinger equation. First, let us consider the one-particle problem. We deal with the two-component spinors placed in the external Electric field directed along the x-axis. We consider the external Electro-magnetic potential in the form: $A_x = Et$. The one-particle Hamiltonian in a subsequent parametrization has the form [9, 8]

$$H = v \left(\begin{array}{cc} 0 & \Big[(\hat{p}_x + Et) - i\hat{p}_y\Big]^J \\ \Big[(\hat{p}_x + Et) + i\hat{p}_y\Big]^J & 0 \end{array} \right). \quad (1)$$

Here v is a constant that is equal to Fermi velocity for the case of monolayer, J is the number of layers. Schrodinger equation has the usual form

$$i\partial_t \Psi = H\Psi. \quad (2)$$

Its solution is

$$\Psi(t) = P \exp \left[-i \int_{t_0}^{t} H(t)dt\right] \Psi(t_0) = \hat{U}(t)\Psi(t_0). \quad (3)$$

Here the path-ordered exponent is used. Operator $\hat{U}(t)$ is unitary by construction. Later on we imply periodic boundary conditions in space-coordinates. That's why Ψ can be decomposed into the sum over the quantized 2-momenta: $\Psi(t,x) = \sum_{p_x,p_y} e^{ip_x x + ip_y y}\psi_p(t)$. For $\psi(t)$ we have:

$$\psi_p(t) = P \exp \left(-iv \int_{t_0}^{t} H[p,t]dt\right) \psi_p(t_0), \quad (4)$$

$$H[p,t] = \left(\begin{array}{cc} 0 & \Big[(p_x + Et) - ip_y\Big]^J \\ \Big[(p_x + Et) + ip_y\Big]^J & 0 \end{array} \right).$$

3. Boundary conditions. In our semiclassical consideration we imply that J is odd. However, analytical

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continuation will allow us to obtain final results for even values of J as well. It is implied that at $t < t_0$ Electric field is absent and we have

$$\psi_p(t) = R[p,t]^{+} e^{-iv|P|^J\sigma_3 t} R[p,t]\psi_p(t_0), \qquad (5)$$
$$R[p,t] = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & \Big(P^*/|P|\Big)^J \\ -\Big(P/|P|\Big)^J & 1 \end{pmatrix},$$

where $P = p_x + Et_0 + ip_y$. Boundary conditions at t_0 must correspond to the negative energy levels occupied:

$$\begin{aligned} 0 &= \psi_1 + \Big(P^*/|P|\Big)^J \psi_2, \\ 1 &= \frac{1}{\sqrt{2}}[\psi_2 - \Big(P/|P|\Big)^J \psi_1]. \end{aligned} \qquad (6)$$

It is supposed that at $t > t_0 + T$ Electric field is switched off again. Then at $t = t_0 + T$ we have

$$\begin{aligned} \eta_+ &= \frac{1}{\sqrt{2}}[\psi_1 + \Big(P^*/|P|\Big)^J \psi_2], \\ \eta_- &= \frac{1}{\sqrt{2}}[\psi_2 - \Big(P/|P|\Big)^J \psi_1], \end{aligned} \qquad (7)$$

the value $|\eta_+|^2$ is the probability that the electron-hole pair has been created, while $|\eta_-|^2$ is the probability that the negative energy level remains occupied.

For the semiclassical consideration it is useful to consider $t_0 = -T/2 \to -\infty$. The essence of semiclassical methodology is the consideration of large frequencies that are in this case $p_x + Et$. That's why we require at $t = t_0 = -T/2$

$$\begin{aligned} 0 &= \psi_1 - \psi_2, \\ 1 &= \frac{1}{\sqrt{2}}(\psi_2 + \psi_1). \end{aligned} \qquad (8)$$

At $t = +T/2$ we denote

$$\begin{aligned} \eta_+ &= \frac{1}{\sqrt{2}}(\psi_1 + \psi_2), \\ \eta_- &= \frac{1}{\sqrt{2}}(\psi_2 - \psi_1), \end{aligned} \qquad (9)$$

where again the value $|\eta_+|^2$ is the probability that the electron-hole pair has been created. This consideration implies $-ET/2 + p_x < 0$ and $ET/2 + p_x > 0$.

For $ET/2 < p_x$ ($p_x < -ET/2$) boundary conditions at $t = t_0 = -T/2$ are:

$$\begin{aligned} 0 &= \psi_1 \pm \psi_2, \\ 1 &= \frac{1}{\sqrt{2}}(\psi_2 \mp \psi_1). \end{aligned} \qquad (10)$$

Here the upper sign is for $ET/2 < p_x$ while the lower one is for $-ET/2 > p_x$. At $t = +T/2$ we expect

$$\begin{aligned} \eta_+ &= \frac{1}{\sqrt{2}}(\psi_1 \pm \psi_2), \\ \eta_- &= \frac{1}{\sqrt{2}}(\psi_2 \mp \psi_1), \end{aligned} \qquad (11)$$

where again the value $|\eta_+|^2$ is the probability that the electron-hole pair has been created.

4. Semiclassical consideration. Let us now introduce the notations:

$$\tau = \Big(\frac{v}{E}\Big)^{1/(J+1)}(p_x + Et), \quad \Pi = \Big(\frac{v}{E}\Big)^{1/(J+1)} p_y,$$
$$\Theta = (\tau + i\Pi)^J. \qquad (12)$$

Then

$$\psi_p(t) = P\exp\Big[-i\int \begin{pmatrix} 0 & \Theta^* \\ \Theta & 0 \end{pmatrix} d\tau\Big]\psi_p(t_0). \qquad (13)$$

The corresponding system of equations at $t > t_0$ is:

$$\begin{aligned} i\psi_1' &= \Theta^*\psi_2, \\ i\psi_2' &= \Theta\psi_1. \end{aligned} \qquad (14)$$

For $\psi_{1,2}$ we have:

$$\psi_2 = i\psi_1'/u^J, \quad u = \tau - i\Pi,$$
$$\Big(\frac{1}{u^J}\partial_u\Big)^2\psi_1 + \Big(1 + \frac{2i\Pi}{u}\Big)\psi_1 = 0. \qquad (15)$$

We introduce new variable $z = u^{J+1}/(J+1)$. The resulting equation is

$$[\partial_z]^2\psi_1 + \Big[1 + \frac{2i\Pi}{(J+1)^{1/(J+1)} z^{1/(J+1)}}\Big]^J \psi_1 = 0. \qquad (16)$$

We represent $\psi_1 = e^{is}$ and obtain the equation for s that is considered iteratively. In the first approximation we neglect the derivatives of s higher than the first derivative. In order to calculate the second approximation we substitute the first approximation to the expression for s'' etc. When only the first and the second terms are kept, the wave functions are given by

$$\begin{aligned} \psi_1 &= c_1\Big(\frac{\tau - i\Pi}{\tau + i\Pi}\Big)^{J/4}\exp\Big[-i\int_{\tau_0}^{\tau}(\tau^2 + \Pi^2)^{J/2}d\tau\Big] + \\ &+ c_2\Big(\frac{\tau - i\Pi}{\tau + i\Pi}\Big)^{J/4}\exp\Big[i\int_{\tau_0}^{\tau}(\tau^2 + \Pi^2)^{J/2}d\tau\Big], \\ \psi_2 &= c_1\Big(\frac{\tau + i\Pi}{\tau - i\Pi}\Big)^{J/4}\exp\Big[-i\int_{\tau_0}^{\tau}(\tau^2 + \Pi^2)^{J/2}d\tau\Big] - \\ &- c_2\Big(\frac{\tau + i\Pi}{\tau - i\Pi}\Big)^{J/4}\exp\Big[i\int_{\tau_0}^{\tau}(\tau^2 + \Pi^2)^{J/2}d\tau\Big]. \end{aligned} \qquad (17)$$

The considered approximation is valid if the second approximation is smaller than the first one. This leads to the condition

$$\left|\frac{J\Pi}{(\tau^2+\Pi^2)^{J/2+1}}\right| = \frac{p_y(E/v)^{J/[2(J+1)]}}{\left[(p_x+Et)^2+p_y^2\right]^{(J+2)/2}} \ll 1. \tag{18}$$

Next, we use the fact that the given semiclassical approximation gives the solution of Eq. (15) not only for large real values of τ but for the complex values of τ with large $|\tau|$. That's why analytical continuation of the solution at $\tau\to-\infty$ gives the solution at $\tau\to+\infty$. (The continuation is performed along the line placed at $|\tau|\to\infty$.)

At $ET/2 > p_x > -ET/2$ boundary conditions give $c_2 = 0$. The probability that the pair is created is $|\eta_+|^2$, where

$$\begin{aligned}\eta_+ &= \exp\left[-i\int_{\tau_0}^{\tau}(\tau^2+\Pi^2)^{J/2}d\tau\right] = \\ &= \exp\left[\frac{|\Pi|^{J+1}}{2}\int(1-z)^{J/2}z^{-1/2}dz\right] = \\ &= \exp\left[-|\Pi|^{J+1}B(J/2+1,1/2)\right], \quad z=-(\tau/\Pi)^2.\end{aligned} \tag{19}$$

Here we consider the contour placed at infinity with the orientation such that $|\eta_+|$ remains less than unity.

We have

$$\begin{aligned}|\eta_+|^2 &= \exp\left[-\alpha\Big(|p_y|/E^{1/(J+1)}\Big)^{J+1}\right], \\ \alpha &= 2\pi v\frac{J!!}{(J+1)!!} = 2vB\left(\frac{J}{2}+1,\frac{1}{2}\right) = \\ &= 2vJB\left(\frac{3}{2},\frac{J}{2}\right).\end{aligned} \tag{20}$$

Written in this form our result can be continued analytically to even values of J. At the same time known results for $J = 1, 2$ are reproduced [4, 8].

At $ET/2 < p_x$ ($p_x < -ET/2$) boundary conditions give $c_1 = 0$ ($c_2 = 0$). In both cases semicassical approximation gives $|\eta_-| = 1$ that means that the electron-hole pair is not created.

Below we check the obtained above value of the probability that the electron-hole pair is created with the given values of momenta (p_x, p_y). We do this in two ways: via the application of the semiclassical approximation in its more classical form and via the consideration of the exact solution of the Schrodinger equation (at $J = 1$).

5. More classical semiclassics. The problem of pair creation can be considered in the gauge $A_0 = Ex$. Then we have stationary Schrodinger equation $H\Psi = \epsilon\Psi$ with

$$H = \begin{pmatrix} Ex & v\Big(\hat{p}_x - i\hat{p}_y\Big)^J \\ v\Big(\hat{p}_x + i\hat{p}_y\Big)^J & Ex \end{pmatrix}. \tag{21}$$

We proceed with the rescaling $z = (E/v)^{1/(J+1)}x$, and $\omega = (1/vE^J)^{1/(J+1)}\epsilon$. Then

$$\begin{aligned}&(z-\omega)\psi_1 + (-i\partial_z - i\Pi)^J\psi_2 = 0, \\ &(z-\omega)\psi_2 + (-i\partial_z + i\Pi)^J\psi_1 = 0.\end{aligned} \tag{22}$$

The first order semiclassical approximation for $\psi_{1,2}$ gives

$$\psi = \exp\left\{\pm i\int[-\Pi^2 + (z-\omega)^{2/J}]^{1/2}dz\right\}. \tag{23}$$

Integration over the classically forbidden region $-\Pi^2 + (z-\omega)^{2/J} < 0$ gives the pair production probability:

$$\begin{aligned}|\eta|^2 &= \exp\left\{-\alpha\Big[p_y/E^{1/(J+1)}\Big]^{J+1}\right\}, \\ \alpha &= 2vJB\Big(\frac{3}{2},\frac{J}{2}\Big).\end{aligned} \tag{24}$$

This expression coincides with the one derived above.

6. Exact solution at $J = 1$. Let us introduce notations $\psi_+ = \psi_1 - \psi_2$ and $\psi_- = \psi_1 - \psi_2$. Then

$$\begin{aligned}\psi_- &= \frac{1}{\Pi}\psi_+' + i\frac{\tau}{\Pi}\psi_+, \\ \psi_+'' &+ (i + \Pi^2 + \tau^2)\psi_+ = 0.\end{aligned} \tag{25}$$

We change variables $\tau = \frac{1}{\sqrt{2}}e^{i\pi/4}z$. Then

$$\begin{aligned}\psi_- &= \frac{\sqrt{2}e^{-i\pi/4}}{\Pi}(\partial_z - z/2)\psi_+, \\ \psi_+'' &+ [1/2 + (i\Pi^2/2 - 1) - z^2/4]\psi_+ = 0.\end{aligned} \tag{26}$$

The solution is

$$\begin{aligned}\psi_+ &= b_1D_{-i\Pi^2/2}(\sqrt{2}e^{-3i\pi/4}\tau) + b_2D_{i\Pi^2/2-1}(\sqrt{2}e^{3\pi i/4}\tau), \\ \psi_- &= \frac{\sqrt{2}e^{-i\pi/4}}{\Pi}[-b_1\Pi^2/2D_{-i\Pi^2/2-1}(\sqrt{2}e^{-3i\pi/4}\tau) + \\ &+ b_2D_{i\Pi^2/2}(\sqrt{2}e^{3i\pi/4}\tau)].\end{aligned} \tag{27}$$

The consideration of usual boundary conditions leads to rather complicated algebra. So we come to the semiclassical boundary conditions ($\tau_0\to+\infty$):

$$\begin{aligned}\sqrt{2} &= b_1D_{-i\Pi^2/2}(\sqrt{2}e^{i\pi/4}\tau_0) + b_2D_{i\Pi^2/2-1}(\sqrt{2}e^{-i\pi/4}\tau_0), \\ 0 &= -b_1\Pi^2/2D_{-i\Pi^2/2-1}(\sqrt{2}e^{i\pi/4}\tau_0) + \\ &+ b_2D_{i\Pi^2/2}(\sqrt{2}e^{-i\pi/4}\tau_0).\end{aligned} \tag{28}$$

Using asymptotic expansion for Weber function we come to $b_2 = 0$ and

$$|\eta_+|^2 = e^{-\pi\Pi^2} = e^{-\pi(v_F/E)p_y^2} \qquad (29)$$

is the probability that the electron-hole pair is created.

7. Field-theoretical consideration. The fact that the particles do not interact with each other allows to reduce the field-theoretical problem to the quantum mechanical one. Namely, we arrive at the following pattern. Modes for different values of momenta propagate independently. At $t \le t_0$ all states with negative values of energy are occupied while all states with positive values of energy are vacant. Their evolution in time is governed by the one-particle Schrodinger equation. At $t = t_0 + T$ the wave function already has the nonzero component corresponding to positive energy. Its squared absolute value is the probability that the electron-hole pair is created.

Let us calculate the probability that vacuum remains vacuum P_v (vacuum persistence probability). According to the above presented calculation this probability is

$$P_v = \Pi_{p_x,p_y}\Big(1 - \exp\{-\alpha[|p_y|/E^{1/(J+1)}]^{J+1}\}\Big)^{g_s g_v} = \\ = e^{-2\mathrm{Im}S}. \qquad (30)$$

Here S is the effective action, the factors $g_s = 2$ and $g_v = 2$ are spin and valley degeneracies. The product is over the momenta that satisfy

$$ET/2 > p_x > -ET/2. \qquad (31)$$

We have (L is the linear size of the graphene sheet):

$$\omega = \frac{2\,\mathrm{Im}S}{TL^2} = -g_s g_v \frac{E}{2\pi L} \times \\ \times \sum_{p_y = \frac{2\pi}{L}K} \log\Big\{1 - \exp[-\alpha(|p_y|/E^{1/2})^{J+1}]\Big\} \approx \\ \approx -g_s g_v \frac{E}{2\pi} \times \\ \times \int \frac{dp_y}{2\pi} \log\Big\{1 - \exp[-\alpha(|p_y|/E^{1/(J+1)})^{J+1}]\Big\} = \\ = g_s g_v \frac{E}{2\pi} \sum_n \frac{1}{n} \times \\ \times \int \frac{dp_y}{2\pi} \exp\{-\alpha n[|p_y|/E^{1/(J+1)}]^{J+1}\} = \\ = g_s g_v \frac{E^{(J+2)/(J+1)}}{2(J+1)\pi^2(\alpha)^{1/(J+1)}} \times \\ \times \zeta\Big(\frac{J+2}{J+1}\Big)\Gamma\Big(\frac{1}{J+1}\Big). \qquad (32)$$

Here α is given by Eq. (20). According to [10] a different quantity is considered as the pair production rate:

$$\Gamma = \langle|\eta_+|^2\rangle/(L^2T) = \\ = g_s g_v \frac{E}{2\pi L} \sum_{p_y = \frac{2\pi}{L}K} \exp[-\alpha(|p_y|/E^{1/2})^{J+1}] = \\ = g_s g_v \frac{E^{(J+2)/(J+1)}}{2(J+1)\pi^2(\alpha)^{1/(J+1)}}\Gamma\Big(\frac{1}{J+1}\Big). \qquad (33)$$

The form of the functional dependence of Γ on E coincides with that of mentioned in [8].

8. Conclusions. In the present paper we calculate the pair production rate and the vacuum persistence probability for the multilayer graphene. We develop the semiclassical technique within the approach used earlier in monolayer graphene. Our method reproduces known results for monolayer and bilayer graphene. Following [10] we consider the single pair creation rate Γ and $\omega = -\log P_v/TL^2$ (where P_v is the vacuum persistence probability) as different quantities. The possibility to consider ω as a production rate of multiple states remains open and requires an additional investigation.

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