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Faces of matrix models

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Partition functions of eigenvalue matrix models possess a number of very different descriptions: as matrix integrals, as solutions to linear and non-linear equations, as τ-functions of integrable hierarchies and as special-geometry prepotentials, as result of the action of W-operators and of various recursions on elementary input data, as gluing of certain elementary building blocks. All this explains the central role of such matrix models in modern mathematical physics: they provide the basic "special functions" to express the answers and relations between them, and they serve as a dream model of what one should try to achieve in any other field.

Matrix model theory [1] studies the integral

$$Z_N(g) \sim \int_{N\times N} dM\, e^{-\frac{1}{2g}\mathrm{tr}\, M^2} = \\ = \int \prod_{i=1}^N dM_{ii} \prod_{i<j}^N d^2 M_{ij} \times \\ \times \exp\left[-\frac{1}{2g}\left(\sum_{i=1}^N M_{ii}^2 + 2\sum_{i<j}^N |M_{ij}|^2\right)\right] \tag{1}$$

over $N\times N$ Hermitian matrices M as a toy-example of quantum field and even string theory. It is spectacular, how much one can learn from this seemingly obvious problem.

What does it mean to study an integral?

First, we can simply take it. In this particular case the answer is simple:

$$Z_N(g) \sim (2\pi g)^{N^2/2} \tag{2}$$

and does not look very interesting. However it only seems so. As usual, of interest is not the answer itself, but its *decomposition*, implied, by internal structure of our "theory". And the more we know about these structures. The more interesting decompositions we can obtain. In this particular case we could notice that $M = UDU^\dagger$, where $D = \mathrm{diag}\{x_i\}$ matrix, made from eigenvalues of M, and U is a unitary matrix. Then the same integral is decomposed into two – over unitary matrix U and over N eigenvalues $\{x_i\}$. Factoring away the volume V_N of the unitary group, we obtain:

$$Z_N = \frac{1}{N!}\int \prod_{i<j}^N (x_i - x_j)^2 \prod_{i=1}^M e^{-x_i^2/2g} dx_i = \\ = \frac{V_1^N}{N! V_N}\int_{N\times N} dM e^{-\frac{1}{2g}\mathrm{tr}\, M^2}. \tag{3}$$

This is already a somewhat non-trivial decomposition, because

$$V_N = \frac{(2\pi)^{N(N+1)/2}}{\prod_{k=1}^N k!} \tag{4}$$

what is a considerably more complicated expression than the original (2).

Second, to study an integral in QFT sense means to treat it as measure, and consider all possible *correlators*. This means that of interest is not the (1) itself, but the averages

$$C_{i_1,\ldots,i_k} = \left\langle \mathrm{tr}\, M^{i_1}\ldots \mathrm{tr}\, M^{i_k}\right\rangle = \\ = \frac{\int \mathrm{tr}\, M^{i_1}\ldots\mathrm{tr}\, M^{i_k} e^{-\frac{1}{2g}\mathrm{tr}\, M^2} dM}{\int e^{-\frac{1}{2g}\mathrm{tr}\, M^2} dM} \tag{5}$$

or even their connected counterparts, like

$$C_{ij}^{\mathrm{conn}} = C_{ij} - C_i C_j. \tag{6}$$

This is already a far-less-trivial problem, and looking at the very first examples one immediately observes an emergency of new structure:

$$C_0(N) = N, \\ C_2(N) = gN^2, \\ C_4(N) = g^2(2N^3 + N) \sim 2(gN)^3 + g^2(gN), \\ C_6(N) = g^3(5N^4 + 2N^2) \sim 5(gN)^4 + 2g^2(gN)^2, \\ \ldots \tag{7}$$

The fact that each correlator is a polynomial (not a monomial) in N is encoded in the idea of *loop expansion*. The fact that all coefficients are integers signals about connection to combinatorics and is encoded in the idea of *topological theories*.

Third, if we move in the direction of string theory, we need not just correlators: we need *generating functions*. For the set of $C_{i_1,\ldots,i_k}$ there are two obvious options:

$$Z\{t\} = \frac{V_1^N}{N!V_N} \times$$
$$\times \int dM \exp\left(-\frac{1}{2g}\mathrm{tr}\, M^2 + \sum_{k=0}^{\infty} t_k \mathrm{tr}\, M^k\right) = e^{F\{t\}} \quad (8)$$

and

$$\rho^{(m)}\{z\} = \left\langle \prod_{i=1}^{m} \mathrm{tr}\, \frac{dz_i}{z_i - M} \right\rangle. \quad (9)$$

Then we have

$$C_{i_1,\dots,i_k} = \frac{1}{Z_N}\frac{\partial^k Z_N}{\partial t_{i_1}\dots\partial t_{i_k}},$$
$$C^{\mathrm{conn}}_{i_1,\dots,i_k} = \frac{\partial^k \log Z_N}{\partial t_{i_1}\dots\partial t_{i_k}} \quad (10)$$

and

$$\rho^{(m)}\{z\} = \frac{1}{Z_N}\prod_{i}^{m}\hat{\nabla}(z_i)Z_N, \quad (11)$$

where $\hat{\nabla}(z) = \sum_{k=0}^{\infty}\frac{dz}{z^{k+1}}\frac{\partial}{\partial t_k}$. One can also introduce the connected resolvent

$$\rho^{(m)}_{\mathrm{conn}}\{z\} = \prod_{i}^{m}\hat{\nabla}(z_i)\log Z_N. \quad (12)$$

The fact that correlators C_I where polynomials in N is now expressed in the *genus expansion* of the free energy and connected resolvents:

$$F\{t|g,N\} = \sum_{p=0}^{\infty} g^{2p-2}F_p\{t|gN\} \quad (13)$$

and similarly

$$\rho^{(m)}_{\mathrm{conn}}\{z\} = \sum_{p=0}^{\infty} g^{2p}\rho^{(p|m)}\{z|gN\}. \quad (14)$$

Already at this stage something highly non-trivial shows up. This becomes clear from a look on the first few resolvents:

$$\rho^{(0|1)}(z) = y(z)dz/2,$$
$$\rho^{(1|1)}(z) = dz/y^5(z),$$
$$\rho^{(0|2)}(z_1,z_2) = \frac{1}{(z_1-z_2)^2}\frac{dz_1dz_2}{y(z_1)y(z_2)},$$
$$\dots \quad (15)$$

They all are meromorphic (poly)differentials on a Riemann surface

$$\Sigma: \quad y^2 = z^2 - 4(gN) \quad (16)$$

which is called the *spectral curve*.

According to the string-theory approach, from this point we should move far enough in a number of different directions.

Other phases. As soon as we introduced the generating function $Z\{t\}$, we can start treating it non-perturbatively. This means that t_k are considered not just as infinitesimal expansion parameters, defining a *germe*, but as the coupling constants, and study what happens when they take finite (or even infinite) values. Then $Z\{t\}$ defines a partition function of a *family* of theories, called *non-perturbative* partition function. This partition function can be re-expanded not only around the Gaussian point, but around any *background potential* $V(M) = \sum_k T_k M^k$. Partition function (particular branch of it) then becomes also a function of parameters T_k, which parameterize the *moduli* of the spectral curve. Phase transitions take place when the genus of the curve changes – it is controlled by the number of extrema of the background potential. The study of these dependencies is the subject of *Seiberg–Witten theory* [2], in matrix-model context the corresponding field is sometime called the theory multi-cut solutions or of the *Dijkgraaf–Vafa phases* [3]. The particular *branch* of partition function is also known as *CIV prepotential* [4]. The most interesting feature of this prepotential are Seiberg–Witten special-geometry equations, describing dependence on the moduli by introducing very special "*flat*" coordinates a_k instead of T_k:

$$\begin{cases} a_k = \oint_{A_k}\Omega, \\ \dfrac{\partial F}{\partial a_k} = \oint_{B_k}\Omega \end{cases} \quad (17)$$

and the role of the Seiberg–Witten differential on the spectral curve is presumably played by the 1-point resolvent $\Omega(z) = \rho^{(1)}(z)$ [5]. The system of interrelated multidensities $\rho_{\Sigma}^{(p|m)}$ can in fact be built in a universal way for arbitrary Seiberg–Witten family of spectral curves Σ – this procedure is now known as AMM/EO topological recursion [6] and has surprisingly many applications. Whenever partition function can be reconstructed in this way, this signals about the matrix-model hidden behind the scene – and there are already numerous examples, when recursion works, but the matrix model is not yet found.

Various limits. Non-perturbative partition function has a huge variety of different limits and critical behaviors in the vicinities of all its numerous singularities. The standard large-N, genus-zero and multiscaling limits are just the examples. Related problem is the study of convergency properties of various perturbative series. All this is very important in applications and constitutes, perhaps, the biggest parts of traditional matrix-model theory.

Other observables. In string-theory paradigm there is no special preference for any *obvious* choice of observables. Instead of the correlators C_I of the monomials $\mathrm{tr}\, M_i$ one could study those, say, of the "Wilson loops" $\mathrm{tr}\left(e^{sM}\right)$, and form many other generating functions, different from (8) and (9), like the celebrated Harer–Zagier exact 1-point function [7]

$$\phi(z|\lambda) = \sum_{N=0}^{\infty} \lambda^N \sum_{k=0}^{\infty} \frac{z^{2k}\left\langle \mathrm{tr} M^{2k}\right\rangle}{(2k-1)!!} = \\ = \frac{\lambda}{(1-\lambda)\Big[(1-\lambda)-(1+\lambda)z^2\Big]} \tag{18}$$

and Brezin–Hikami integrals [8]

$$\left\langle \prod_{i=1}^{k} \mathrm{tr}(e^{s_i M}) \right\rangle = \prod_{i=1}^{k} \frac{e^{s_i^2/2}}{s_i} \oint e^{u_i s_i} du_i \left(1+\frac{s_i}{u_i}\right)^N \times \\ \times \prod_{i<j}^{k} \frac{(u_i-u_j)(u_i-u_j+s_i-s_j)}{(u_i-u_j+s_i)(u_i+u_j-s_j)}. \tag{19}$$

The number of integrals here is k, not N, as in (3). In fact, these two subjects are unexpectedly closely related [9]. Harer–Zagier functions capture contributions from all genera – they differ from (8) by a kind of Pade transform and allow to put under the control the divergence of perturbative genus expansion. Instead they hide all the information related to spectral curves and Seiberg–Witten equations – but are capable to provide a closed expression for the Seiberg–Witten differential $\Omega(z) = \rho^{(1)}(z)$. Unfortunately, they are much more difficult to study than the resolvents.

Alternative formulations. For non-perturbative partition functions integrals (be they matrix or functional) provide only a description of particular phases: or, in worst case just the perturbative *germes* at particular points. More adequate are formulations in terms of *D-modules* or τ-functions, characterizing partition functions as solutions to linear or quadratic equations respectively. It is still unclear, how general is the existence of quadratic (integrability-theory) structures and if higher non-linearities can also be relevant. At this moment, the "*matrix-model τ-functions*" – usually, KP/Toda-functions, satisfying also a linear *string equation*, and, as a corollary, a whole infinite set of linear "Virasoro constraints" [10] are the most profound special functions, encountered in modern mathematical physics. They are natural for presentation of quantitative results in various fields of string theory, and their investigation is one of the primary purposes of modern science.

Integrability and W-representation. Emergency of non-linear (integrable) relations, like [11]

$$\frac{\partial^2 \log Z_N}{\partial t_1^2} = \frac{Z_{N+1}Z_{N-1}}{Z_N^2} \tag{20}$$

for (8), is so non-trivial and so universal in string theory, that it can be considered as one of the main features of non-perturbative physics – still very mysterious. One should look for adequate ways to characterize these structures. Non-trivial τ-functions can be made from the "trivial" ones by integrability-preserving transforms, described in terms of the W-operators, which move the points in the Universal Grassmannian, parameterizing the space of the KP/Toda (free-fermion) τ-functions. In other words, a matrix-model τ-function can be considered as a result of the "evolution", driven by cut-and-join (W) operators from some simple "initial conditions" [12]:

$$Z\{t\} = e^{\hat{W}}\tau_0\{t\}. \tag{21}$$

For (8) this W-representation looks as follows:

$$Z_N\{t\} = \exp\left\{\sum_{a,b}\left[a t_a b t_b \frac{\partial}{\partial t_{a+b-2}} + \right.\right. \\ \left.\left. +(a+b+2)t_{a+b+2}\frac{\partial^2}{\partial t_a \partial t_b}\right]\right\} e^{Nt_0}. \tag{22}$$

Generalizations. According to string-theory paradigm, one should not just embed original model in a set the similar ones by exponentiating all naive observables, one should also deform everything else, including the discrete parameters. In application to matrix models this means that starting from (1) one should not just switch from quadratic to arbitrary potential, not just treat N as one of parameters, but also substitute Hermitian matrices by others: unitary, orthogonal, symplectic, belonging to exceptional and other Lie algebras, to generic tensorial categories etc. Of all this the most far-going so far are extensions to unitary matrix models [13] and to β-ensembles [14]. In all cases one expects to find all the relevant representations: not only through traditional integral formulas, but also as D-modules, as τ-functions, through W-operators, through topological recursion which start from peculiar spectral curves, through Harer–Zagier-type recursions. Some results in these directions exist, but they are far from being exhaustive.

External fields and dualities. Another generalization is inclusion of external fields. The simplest possibility is to switch from (1) to

$$\mathcal{Z}(\nu|A)=e^{-\frac{g}{2}\mathrm{tr}A^2}\int_{n\times n} e^{-\frac{1}{2g}\mathrm{tr}M^2+\mathrm{tr}\,MA}(\det M)^{\nu}dM. \quad (23)$$

Determinant is introduced here to make the dependence on A non-trivial, and we also changed the notation for the size of the integration matrix. This is done on purpose, because if this function is considered as a function of the variable $p_k = \mathrm{tr}\,A^{-k}$, it is actually independent of n. In this way one defines *Kontsevich matrix models*, eq. (23) is the Gaussian one, for properties of generic Kontsevich models see [15]. Really remarkable is the duality between (23) and (8):

$$\mathcal{Z}(N|A) \sim Z_N\{t_k\} \quad (24)$$

provided $t_k = \frac{1}{k}p_k = \frac{1}{k}\mathrm{tr}\,A^{-k}$. In fact, this duality [16] can be used in the derivation of Brezin–Hikami formulas (19), which, in turn have non-trivial generalization [17, 9] to at least the cubic Kontsevich model.

Unification. Duality between Gaussian Hermitian and Kontsevich models is just an example of interrelation between two *a priori* different matrix models. The goal of string theory is to unify in a similar way all quantum field theories, and in particular, this applies to unification of all matrix models. Unification does not mean *solving* – that problem belongs to the field of non-linear algebra [18], which studies formulas like

$$\iint dxdy\ e^{ax^2+bxy+dy^2} \sim \frac{1}{\sqrt{4ad-b^2}} = D_{2|2}^{-1/2},$$
$$\iint dxdy\ e^{ax^3+bx^2y+cxy^2+dy^3} \sim D_{2|3}^{-1/6},$$
$$D_{2|3} = 27a^2d^2 - b^2c^2 - 18abcd + 4ac^3 + 4b^3d \quad (25)$$

(in general ordinary discriminants $D_{N|r}$ control singularities of integral discriminants). Unification means that all seemingly different non-perturbative partition functions either are interrelated (by dualities), or are all reductions of some larger partition function (arise at particular loci in the extended space of time-variables), or are all composed from some elementary building blocks. It turns out that the last, most promising, possibility can be true, at least in the world of the eigenvalue matrix models. Namely, at least all the Dijkgraaf–Vafa partition functions can be obtained by a universal gluing procedure from a few basic elements [19]:

$$Z\{t\} = e^{\hat{U}}\prod_{i=1}^{k} Z^{(i)}\{t^{(i)}\}, \quad (26)$$

where $\hat{U}$ is bilinear in derivatives over $t^{(i)}$-variables. This gluing procedure is closely related to AMM/EO recursion [6] and can be considered as one of its most profound implications. The role of the elementary building blocks $Z^{(i)}$ play several important matrix models which posses a sphere with punctures as their spectral curves: the Gaussian Hermitian model, the cubic Kontsevich model and the Brezin–Gross–Witten model [20].

Applications. Matrix model theory has infinitely many applications in all branches of science, far beyond pure mathematics, string theory and even physics. Still, it deserves mentioning a few relatively new examples, concerning the abstract fields of research, in order to illustrate once again the influence of matrix model intuition on our understanding of basic problems. These recent applications also emphasize the role of the *character calculus* – one of the most important matrix-model-theory technical methods. Moreover, matrix models themselves are not present very explicitly, what are discovered are the typical structures and relations, pertinent for matrix-model partition functions.

The first subject is **Hurwitz theory** [21]. Today it is clear that this is basically the story about the algebra of cut-and-join operators, which are well known in matrix model theory

$$\hat{W}_R = :\prod_i \mathrm{tr}\left(M\frac{\partial}{\partial M^{tr}}\right)^{r_i}: . \quad (27)$$

They are labeled by Young diagrams $R = \{r_1 \geq r_2\}$ and have Schur functions (the $GL(\infty)$ characters) $\chi_Q[M]$ as common eigenfunctions:

$$\hat{W}_R\chi_Q[M] = \varphi_Q(R)\chi_Q[M] \quad (28)$$

while eigenvalues $\varphi_Q(R)$ depend on a pair of Young diagrams and are essentially the characters of symmetric group $S(\infty)$. The Hurwitz partition functions describe the sums like

$$\sum_Q d_Q^{2-2p}\varphi_Q(R_1)\ldots\varphi_Q(R_k) \longrightarrow$$
$$\longrightarrow \sum_Q d_Q^2 \exp\left[\sum_R t_R\varphi_Q(R)\right] \quad (29)$$

and possess many properties, typical for matrix-model τ-functions, including the deeply hidden Virasoro-constraints, as well as numerous non-trivial generalizations, involving non-commutative "open-string" algebra, extending the commutative "closed-string" one formed by the $\hat{W}_R$. See [22] for details and references.

The second subject is the **AGT conjecture** [23]: the celebrated identity between $2d$ conformal blocks and Nekrasov expansions [24] of the LMNS functions [25], describing instanton expansions of $4/5/6d$ SYM theories. This subject brings together conformal field theory, Seiberg–Witten theory, classical and quantum integrable systems [26]. At the core of the story is the special "conformal" matrix model [27], which realizes Dotsenko–Fateev representation of conformal blocks, and LMNS functions appear from Selberg integrals, arising in the character expansion of the model. In this language the AGT relation reduces to the Hubbard–Stratanovich duality [28], this works perfectly at $\beta = 1$, but generalization to β-ensembles remains subtle [5]. What is extremely important in this story is that the averages of characters are again characters – and matrix models with this special property seem to become more and more distinguished in modern applications.

The third example is the modern **theory of knots** [29, 30], which studies extended [31] HOMFLY $H_R^{\mathcal{B}}\{p_k|\,q\}$ [32] and superpolynomials $P_R^{\mathcal{B}}\{p_k|\,q,t\}$ [33]. A very interesting matrix model realization here is long known for the underlying Chern–Simons theory, but its generalization in the presence of non-trivial knots is so far available only for torus knots [34] and for $t = q$. In general one expects that the model exists, the measure depends on the braid realization $\mathcal{B}$, and the HOMFLY polynomial in representation R is an average of the $SL(N)$ character:

$$H_R^{\mathcal{B}} = \Big\langle \chi_R[U] \Big\rangle_{\mathcal{B}}. \tag{30}$$

Like in the case of AGT relation, one expects that with this measure the averages of characters will be again simply re-expanded in characters, and such model will be a useful tool to study the character expansions of HOMFLY and superpolynomials, which are responsible for the fast progress in the field in recent months. This is indeed the case for the torus knot $[m,n]$: the measure is given by [34]

$$\Big\langle \ldots \Big\rangle_{[m,n]} = \prod_{i=1}^{N} \int e^{-u_i^2/mng} du_i \times \\ \times \prod_{i<j}^{N} \sinh \frac{u_i - u_j}{m} \sinh \frac{u_i - u_j}{n} \Big(\ \ldots \ \Big) \tag{31}$$

and $\langle \chi_R[U] \rangle_{[m.n]} \sim \chi_R\{kt_k = [kN]_q/[k]_q\}$, moreover, like with all Selberg-type integrals, this property persists for bilinear combinations of characters.

The forth example, which deserves mentioning is a very similar Chern–Simons type matrix-model representation in the very important **ABJM theory** [35], describing N copies of $M2$ branes. The only additional complication is that cosh factors are also present in denominators of the Vandermonde determinants. Despite this complication the model was completely solved in [36] at vanishing times, and the required non-trivial behavior $\sim N^{3/2}$ (instead of the usual $\sim N^2$) of the free energy was reproduced in the large-N limit.

Note that adequate introduction of time variables, suitable for revealing the linear and non-linear relations – in the form of Virasoro constraints and KP/Toda integrability respectively – remains a largely unsolved problem in all these examples, despite there are already many signals, that these or very similar structures should exist. It is one of the primary tasks of matrix-model theory to study and resolve these mysteries.

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