Pis'ma v ZhETF, vol. 96, iss. 9, pp. 627 – 631 © 2012 November 10

# Pion form factor in the range $-10 \le s \le 1$ GeV$^2$

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Submitted 25 September 2012

Based on the field-theory-inspired approach, a new expression for the pion form factor $F_\pi$ is proposed. It takes into account the pseudoscalar meson loops $\pi^+\pi^-$ and $K\bar{K}$ and the mixing of $\rho(770)$ with heavier $\rho(1450)$ and $\rho(1700)$ resonances. The expression possesses correct analytical properties and describes the data in the wide range of the energy squared $-10 \le s \le 1$ GeV$^2$ without introducing the phenomenological Blatt–Weisskopf range parameter $R_\pi$.

The electromagnetic form factor of the pion $F_\pi$ is an important characteristic of the low energy phenomena in particle physics related with the hadronic properties of the electromagnetic current in the theoretical scheme of the vector dominance model [1–3]. There are a number of expressions for this quantity used in the analysis of experimental data in the time-like range $s > 0$. Hereafter $s$ is the center-of-mass-energy squared. The simplest approximate vector dominance model expression based on the effective $\gamma - \rho$ coupling $\propto \rho_\mu A_\mu$ [3],

$$F_\pi(s) = \frac{m_\rho^2 g_{\rho\pi\pi}/g_\rho}{m_\rho^2 - s - i\sqrt{s}\Gamma_{\rho\pi\pi}(s)}, \qquad (1)$$

does not possess the correct analytical properties upon the continuation to the unphysical region $0 \le s < 4m_\pi^2$ and further to the spacelike region $s \le 0$, nor does it takes into account the mixing of the isovector $\rho$-like resonances. Since, phenomenologically [4],

$$\frac{g_{\rho\pi\pi}}{g_\rho} = \left(\frac{9m_\rho\Gamma_{\rho\pi\pi}\Gamma_{\rho ee}}{2\alpha^2 q_\pi^3}\right)^{1/2} \approx 1.20 \qquad (2)$$

the correct normalization $F_\pi(0) = 1$ is satisfied by Eq. (1) only approximately, provided $\Gamma_{\rho\pi\pi}(s)$ vanishes at $s \le 4m_\pi^2$. Hereafter, $\alpha = 1/137$ stands for the fine structure constant. The formula of Gounaris and Sakurai [5] respects the normalization condition and has correct analytical properties. However, being based on some sort of effective radius approximation for the single $\rho(770)$ resonance, it is not suited for taking into account the mixing of $\rho(770)$ with heavier isovector mesons. The expression based on the gauge invariant $\gamma - \rho$ coupling $\propto \rho_{\mu\nu}F_{\mu\nu}$,

$$F_\pi(s) = 1 + \frac{s g_{\rho\pi\pi}/g_\rho}{m_\rho^2 - s - i\sqrt{s}\Gamma_{\rho\pi\pi}(s)}, \qquad (3)$$

respects the correct normalization, but does not have correct analytical properties and breaks unitarity. The earlier expression [6] for $F_\pi$ takes into account the strong isovector mixing, but the normalization is satisfied within the accuracy 20%.

On the other hand, since the pioneer works [7, 8], the space-like region is considered as the test ground of the perturbative QCD predictions for $F_\pi$ addressing, in particular, the issues such as where the QCD asymptotic starts.

In the present work the expression for the pion form factor is obtained which has correct analytical properties in both the space-like and time-like domains and takes into account the mixing of $\rho(770)$ with the heavier resonances $\rho(1450)$ and $\rho(1700)$. Remarkably, it does require the phenomenological Blatt–Weisskopf range factor $R_\pi$. By restricting the consideration to the pseudoscalar meson loops $PP = \pi^+\pi^-$, $K\bar{K}$ admitting the analytical treatment and valid at energies below 1 GeV, the new expression is found and compared with the existing data on $F_\pi$ collected with the detectors SND [9], CMD-2 [10], KLOE [11], and BaBaR [12] at $4m_\pi^2 \le s \le 1$ GeV$^2$. With the resonance parameters found in this region the continuation to the region $-10$ GeV$^2$ $\le s \le 0$ is made and compared with the existing experimental data [13–16].

The new expression for the pion form factor is represented in the form

$$F_\pi(s) = (g_{\gamma\rho_1}, g_{\gamma\rho_2}, g_{\gamma\rho_3})G^{-1}\begin{pmatrix} g_{\rho_1\pi\pi} \\ g_{\rho_2\pi\pi} \\ g_{\rho_3\pi\pi}\end{pmatrix} + \\ + \frac{g_{\gamma\omega}\Pi_{\rho_1\omega}}{D_\omega\Delta}(g_{11}g_{\rho_1\pi\pi} + g_{12}g_{\rho_2\pi\pi} + \\ + g_{13}g_{\rho_3\pi\pi}). \qquad (4)$$

It takes into account both the strong isovector $\rho(770)-\rho(1450)-\rho(1700)$ mixing and the small

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$\rho(770)-\omega(782)$ one, automatically respects the current conservation condition $F_\pi(0)=1$, and possesses correct analytical properties over entire $s$ plane. The notations are as follows. $\rho_1\equiv\rho(770)$, $\rho_2\equiv\rho(1450)$, $\rho_3\equiv\rho(1700)$;

$$G=\begin{pmatrix} D_{\rho_1} & -\Pi_{\rho_1\rho_2} & -\Pi_{\rho_1\rho_3} \\ -\Pi_{\rho_1\rho_2} & D_{\rho_2} & -\Pi_{\rho_2\rho_3} \\ -\Pi_{\rho_1\rho_3} & -\Pi_{\rho_2\rho_3} & D_{\rho_3} \end{pmatrix}$$

is the matrix of inverse propagators, $g_{ij}$ are its matrix elements multiplied by $\Delta = \det G$; $g_{\gamma V} = m_V^2/g_V$, where $g_V$ enters the leptonic partial widths like $\Gamma_{V\to e^+e^-} = 4\pi\alpha^2 m_V/3g_V^2$; $D_{\rho_i} = m_{\rho_i}^2 - s - \Pi_{\rho_i\rho_i}$. The diagonal polarization operators are $\Pi_{\rho_i\rho_i} = g^2_{\rho_i\pi\pi}\left[\Pi(s,m^2_{\rho_i},m^2_\pi)+\frac{1}{2}\Pi(s,m^2_{\rho_i},m^2_K)\right]$, the non-diagonal ones are expressed through the diagonal one:

$$\Pi_{\rho_1\rho_{2,3}} = \frac{g_{\rho_{2,3}\pi\pi}}{g_{\rho_1\pi\pi}}\Pi_{\rho_1\rho_1},$$
$$\Pi_{\rho_2\rho_3} = \frac{g_{\rho_2\pi\pi}g_{\rho_3\pi\pi}}{g^2_{\rho_1\pi\pi}}\Pi_{\rho_1\rho_1} + sa_{23}. \qquad (5)$$

The quantity $a_{23}$ is free parameter. The $\omega(782)$ propagator is $D_\omega = m_\omega^2 - s - i\sqrt{s}\Gamma_\omega(s)$, where $\Gamma_\omega(s)$ includes the $3\pi$ and radiative decay modes, $\Pi_{\rho_1\omega} = \frac{s}{m^2_\omega}\Pi'_{\rho_1\omega} + i\sqrt{s}\left[\frac{1}{3}\Gamma_{\omega\pi\gamma}(s)+3\Gamma_{\omega\eta\gamma}(s)\right]$ is responsible for the $\rho(770)-\omega(782)$ mixing.

The expression for the polarization operator of the vector meson $V$ is obtained from the dispersion representation:

$$\frac{\Pi^{(PP)}_{VV}(s)}{s} = \frac{g^2_{VPP}}{6\pi^2}\int_{4m_P^2}^{\infty}\frac{q^3_{PP}(s')}{s'^{3/2}(s'-s-i\varepsilon)}ds'. \qquad (6)$$

Remaining logarithmic divergence cancels after subtracting the real part of the above expression at $s=m_V^2$. The resulting expression is represented in the form $\Pi(s,m_V^2,m_P^2)=\Pi_0+\Pi_1$, where

$$\Pi_0 = \frac{s}{48\pi^2}\left[8m_P^2\left(\frac{1}{m_V^2}-\frac{1}{s}\right)+v_P^3(m_V^2)\times\right.$$
$$\times\ln\frac{1+v_P(m_V^2)}{1+v_P(m_V^2)}\theta(m_V-2m_P)-$$
$$\left.-\,2\bar v_P^3(m_V^2)\arctan\frac{1}{\bar v_P}\theta(2m_P-m_V)\right],$$
$$\Pi_1 = \frac{sv_P^3(s)}{48\pi^2}\left[i\pi-\ln\frac{1+v_P(s)}{1-v_P(s)}\right]\theta(s-4m_P^2)+$$
$$+\,2\bar v_P^3(s)\arctan\frac{1}{\bar v_P(s)}\theta(4m_P^2-s)\theta(s)-$$
$$-\,v_P^3(s)\ln\frac{v_P(s)+1}{v_P(s)-1}\theta(-s), \qquad (7)$$

and $v_P(s)=\sqrt{1-4m_P^2/s}$, $\bar v_P(s)=\sqrt{4m_P^2/s-1}$, $\theta$ is the step function. The necessary details of the derivation and parametrization are given elsewhere [17].

The quantity to fit is the bare cross section

$$\sigma_{\text{bare}} = \frac{8\pi\alpha^2}{3s^{5/2}}|F_\pi(s)|^2q_\pi^3(s)\left[1+\frac{\alpha}{\pi}a(s)\right], \qquad (8)$$

where $F_\pi(s)$ is given by Eq. (4), $q_\pi(s)=\sqrt{s}v_\pi(s)/2$ is the momentum of the final pion, and the function $a(s)$ allows for the radiation of a photon by the final pions in the point-like approximation [18–21]. The masses of the heavier vector mesons a kept fixed: $m_{\rho_2} = 1450$ MeV, $m_{\rho_3} = 1700$ MeV. The set of free parameters is $m_{\rho_1}$, $g_{\rho_1\pi\pi}$, $g_{\rho_1}$, $m_\omega$, $g_\omega$, $\Pi'_{\rho_1\omega}$, $g_{\rho_2\pi\pi}$, $g_{\rho_2}$, $g_{\rho_3\pi\pi}$, and $a_{23}$. Their obtained values, found from fitting the bare cross section Eq. (8), side-by-side with the corresponding $\chi^2$ per number of degrees of freedom, are listed in Table separately for the four independent measurements of SND [9], CMD-2 [10], KLOE [11], and the BaBaR data [12] restricted to the low-energy range $\sqrt{s}\le 1$ GeV. The bare cross section evaluated with the parameters of Table is compared with the SND [9], CMD-2 [10], KLOE [11], and BaBaR [12] data shown

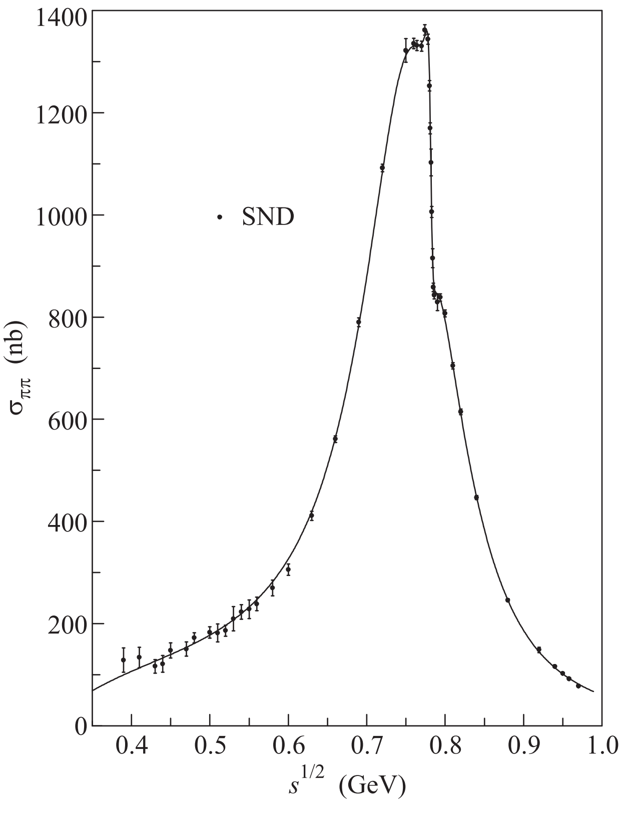


Fig. 1. The bare cross section, Eq. (8), calculated with the resonance parameters obtained from fitting the SND data [9] listed in Table

**The resonance parameters found from fitting the data SND [9], CMD-2 [10], KLOE10 [11], and the BaBaR data [12] restricted to the energies $\sqrt{s} \leq 1$ GeV**

| Parameter | SND | CMD-2 | KLOE10 | BaBaR |
|---|---|---|---|---|
| $m_{\rho_1}$, MeV | $773.76 \pm 0.21$ | $774.70 \pm 0.26$ | $774.36 \pm 0.12$ | $773.92 \pm 0.10$ |
| $g_{\rho_1\pi\pi}$ | $5.798 \pm 0.006$ | $5.785 \pm 0.008$ | $5.778 \pm 0.006$ | $5.785 \pm 0.004$ |
| $g_{\rho_1}$ | $5.130 \pm 0.004$ | $5.193 \pm 0.006$ | $5.242 \pm 0.003$ | $5.167 \pm 0.002$ |
| $m_\omega$, MeV | $781.76 \pm 0.08$ | $782.33 \pm 0.06$ | $782.94 \pm 0.11$ | $782.04 \pm 0.10$ |
| $g_\omega$ | $17.13 \pm 0.30$ | $18.43 \pm 0.47$ | $18.27 \pm 0.45$ | $17.05 \pm 0.29$ |
| $\Pi'_{\rho_1\omega}$, $10^3$ GeV$^2$ | $4.00 \pm 0.07$ | $3.97 \pm 0.10$ | $3.98 \pm 0.09$ | $4.00 \pm 0.06$ |
| $g_{\rho_2\pi\pi}$ | $0.71 \pm 0.35$ | $0.79 \pm 0.26$ | $0.019 \pm 0.004$ | $0.21 \pm 0.04$ |
| $g_{\rho_2}$ | $8.0 \pm 4.4$ | $7.6 \pm 3.4$ | $0.22 \pm 0.07$ | $4.0 \pm 1.0$ |
| $g_{\rho_3\pi\pi}$ | $0.20^{+1.20}_{-0.17}$ | $0.76 \pm 0.75$ | $0.055^{+0.088}_{-0.043}$ | $0.011^{+0.479}_{-0.007}$ |
| $a_{23}$ | $0.002 \pm 0.011$ | $-0.016 \pm 0.057$ | $-0.014 \pm 0.040$ | $-0.0005 \pm 0.0009$ |
| $\chi^2/N_{\rm d.o.f}$ | 54/35 | 34/19 | 87/65 | 216/260 |
| $r_\pi$[fm] | $0.635 \pm 0.054$ | $0.646 \pm 0.059$ | $0.668 \pm 0.039$ | $0.668 \pm 0.053$ |

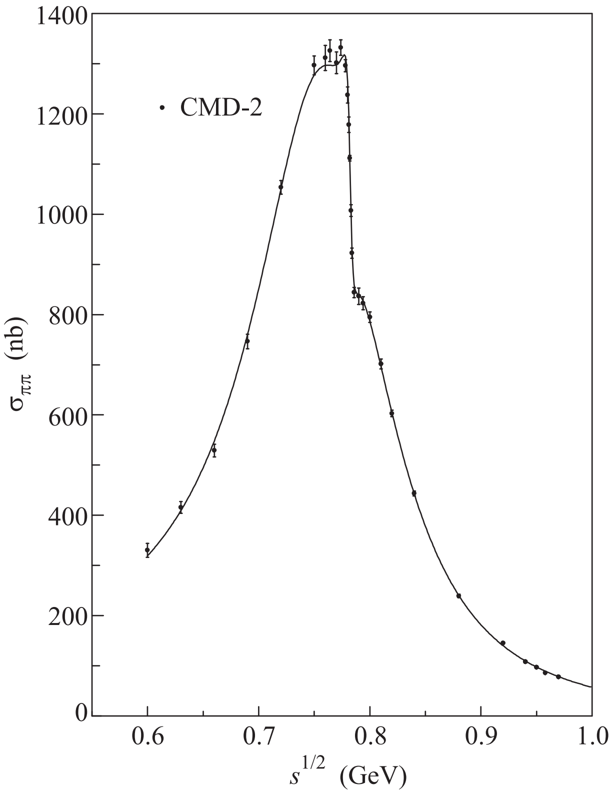


Fig. 2. The same as in Fig. 1, but evaluated with the parameters obtained from fitting the CMD-2 data [10]

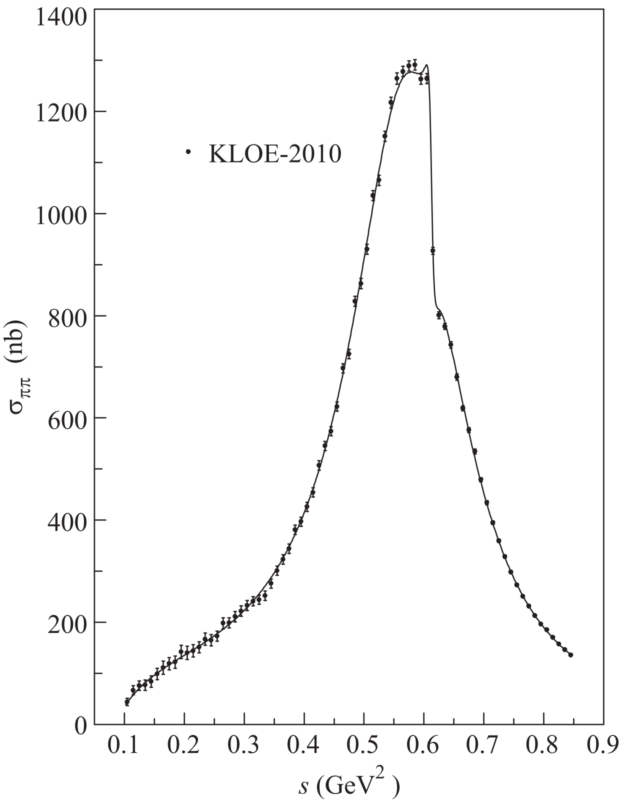


Fig. 3. The same as in Fig. 1, but evaluated with the parameters obtained from fitting the KLOE-2010 data [11]

in Figs. 1, 2, 3, and 4, respectively. The bottom line of this Table shows the values of the pion charge radius, $r_\pi = \sqrt{6\frac{dF_\pi(s)}{ds}}|_{s\to 0}$, calculated with the resonance parameters listed in the Table. For comparison, the averaged value of the pion charge radius cited by the PDG [4] is $r_\pi = 0.672 \pm 0.008$ fm.

An important check of the expression for the pion form factor Eq. (4) and the consistency of the fits is the continuation to the space-like region $s < 0$ accessible in the scattering processes. To this end, one should take the expression for $F_\pi(s)$ at $s < 0$. Having in mind that the $\rho(770)-\omega(782)$ mixing in the region $s < 0$ is negligibly small one can calculate $F_\pi(s)$ in this region.

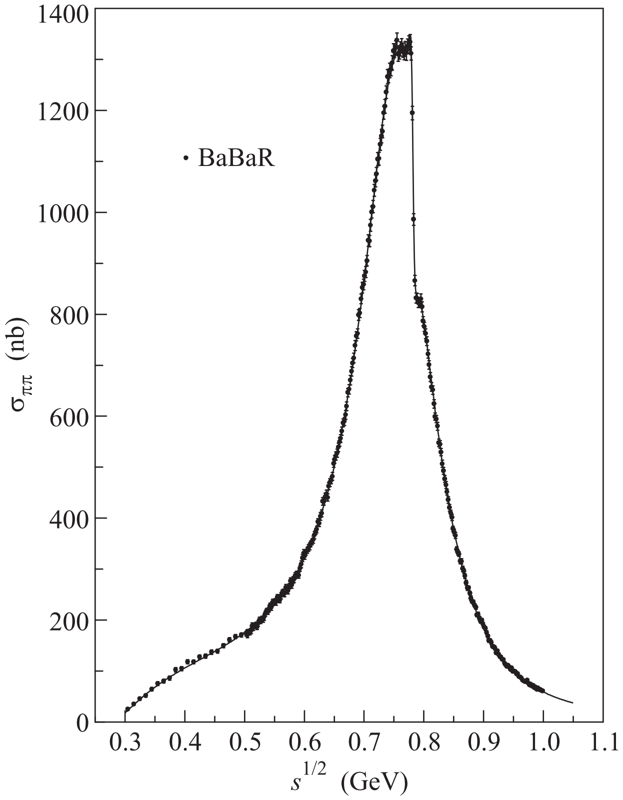


Fig. 4. The same as in Fig. 1, but evaluated with the parameters obtained from fitting the BaBaR data [12] restricted to the energies $\sqrt{s} \leq 1$ GeV

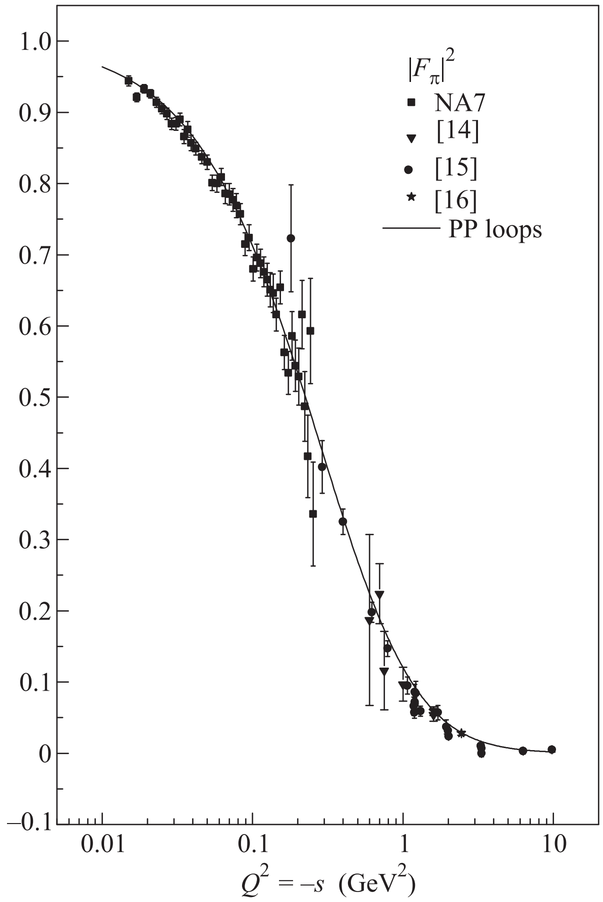


Fig. 5. The pion form factor squared in the space-like region $s < 0$ evaluated using the resonance parameters of the Table, the BaBaR column. The experimental data are: NA7 [13], Bebek et al. [14], Horn et al. [15], Tadevosyan et al. [16]

The results are shown in Fig. 5, where the comparison with the data [13–16] is presented in the case of the resonance parameters found from fitting the BaBaR data [12]. As for the curves corresponding to the parameters found from fitting SND, CMD-2, and KLOE data, they coincide, within the errors, with the curve shown and hence are not drawn in Fig. 5. We emphasize that the data [13–16] are not included to the fits. Hence, a good agreement demonstrated in Fig. 5 makes the evidence in favor of the validity of Eq. (4) for the pion form factor.

Note that the above treatment does not require the commonly accepted Blatt–Weisskopf centrifugal factor $(1 + R_\pi^2 k_R^2)/(1 + R_\pi^2 k^2)$, where $k$ is the pion momentum, in the expression for $\Gamma_{\rho\pi\pi}(s)$ [4]. The fact is that the usage of $R_\pi$ dependent centrifugal barrier penetration factor in particle physics (for example, in the case of the $\rho(770)$ meson [4]), results in the problem which is overlooked. Indeed, the meaning of $R_\pi$ is that this quantity is the characteristic of the potential (or the $t$-channel exchange in field theory) resulting in the phase $\delta_{bg}$ of the potential scattering in addition to the resonance phase [22]. For example, in case of the $P$-wave scattering in the potential $U(r) = G\delta(r - R_\pi)$, where the resonance scattering is possible, the background phase is $\delta_{bg} = -R_\pi k + \arctan(R_\pi k)$. At the usual value of $R_\pi \sim 1$ fm, $\delta_{bg}$ is not small. However, in the $\rho$ meson region, the background phase shift $\delta_{bg}$ is negligible and the phase shift $\delta_1^1$ is completely determined by the resonance. See Fig. 6, where shown are the phase $\delta_1^1$ calculated with the parameters from the BaBaR column of the Table and the data points from Refs. [23, 24] is presented. Therefore, the descriptions of the hadronic resonance distributions taking into account the parameter $R_\pi$ have a dubious character.

To conclude, the new expression, Eq. (4), for the pion form factor $F_\pi(s)$ is obtained which gives a good description of the data of SND, CMD-2, KLOE, BaBaR on $\pi^+\pi^-$ production in $e^+e^-$ annihilation at $\sqrt{s} < 1$ GeV, describes the scattering kinematical domain, and does not contradict the data on $\pi\pi$ scattering phase $\delta_1^1$. Going to higher energies demands the inclusion of the vec-

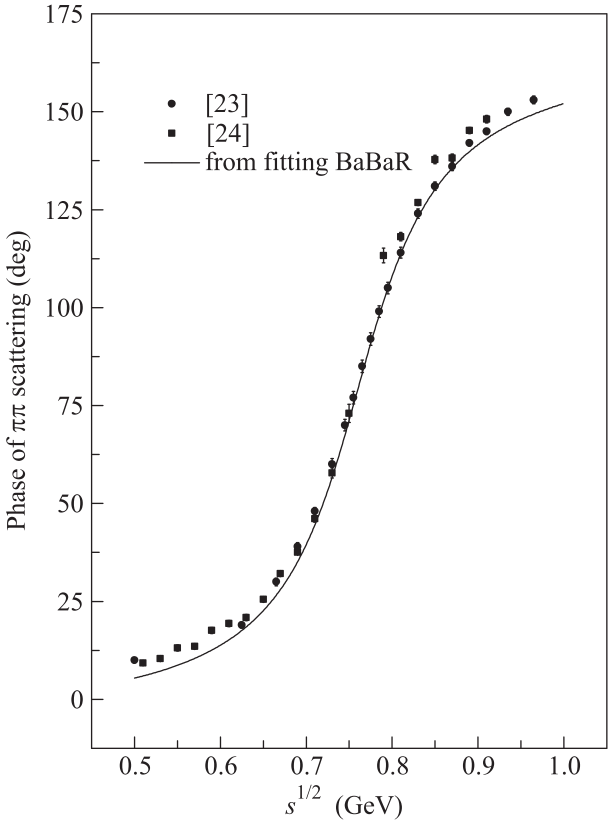


Fig. 6. The phase shift $\delta_1^1$ of $\pi\pi$ scattering. The data are, respectively, Protopopescu et al. [23] and Estabrooks et al. [24]. The curves corresponding to the parameters obtained from fitting the SND, CMD-2, and KLOE data are not shown because they coincide with the curve evaluated using the parameters from the fit of the BaBaR data, shown here

tor – pseudoscalar and axial-vector – pseudoscalar loops. This work is now at progress.

We are grateful to M.N. Achasov for numerous discussions which stimulated the present work. The work is supported in part by Russian Foundation for Basic Research (Grant RFFI # 10-02-00016) and the Interdisciplinary project # 102 of Siberian Division of Russian Academy of Sciences.


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