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Spectral duality in integrable systems from AGT conjecture

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We describe relationships between integrable systems with N degrees of freedom arising from the AGT conjecture. Namely, we prove the equivalence (spectral duality) between the N-cite Heisenberg spin chain and a reduced gl_N Gaudin model both at classical and quantum level. The former one appears on the gauge theory side of the AGT relation in the Nekrasov–Shatashvili (and further the Seiberg–Witten) limit while the latter one is natural on the CFT side. At the classical level, the duality transformation relates the Seiberg–Witten differentials and spectral curves via a bispectral involution. The quantum duality extends this to the equivalence of the corresponding Baxter–Schrödinger equations (quantum spectral curves). This equivalence generalizes both the spectral self-duality between the 2×2 and $N\times N$ representations of the Toda chain and the famous AHH duality.

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In this paper we study the AGT (Alday–Gaiotto–Tachikawa) correspondence [1] at the level of integrable systems [2–4] (see also [5–15]). More exactly, we deal with the AGT inspired models which emerge in the limiting case. The full AGT correspondence associates the conformal block of the Virasoro or W-algebra in two-dimensional conformal field theory with the LMNS (Losev–Moore-Nekrasov–Shatashvili) integral [16] (Nekrasov functions [17]) describing the two-parametric deformation of Seiberg–Witten theory by Ω-background. The classical integrable systems emerge when both deformation parameters are brought to zero, while when only one of the parameters going to zero (the Nekrasov–Shatashvili limit [3]) the integrable system gets quantized. We shall study here only the correspondence between AGT inspired integrable systems in these two limiting cases.

It is important that the two sides of the AGT correspond to *a priori* different types of integrable models which should actually coincide due to AGT. This leads to non-trivial predictions of equivalence of different models and also illuminates what the equivalence exactly means. Here we consider the simplest example of this kind: the equivalence of the four-point conformal block and the prepotential in the $SU(N)$ SUSY theory with vanishing β-function. On the gauge theory side the (classical) integrable system is known [18] to be the Heisenberg chain [19] which is described by the spectral curve (hypersurface in $\mathbb{C}\times\mathbb{C}^*$) $\Gamma^{\mathrm{Heisen}}(w,x)$: $\det[w-T(x)]=0$ with GL_2-valued N-site transfer-matrix $T(x)$ and Seiberg–Witten [20] (SW) differential $dS^{\mathrm{Heisen}}(w,x)=x\frac{dw}{w}$. On the CFT side the corresponding integrable system was argued to be some special reduced Gaudin model [21] defined by its spectral curve $\Gamma^{\mathrm{Gaudin}}(y,z):\det[y-L(z)]=0$ with gl_N-valued Lax matrix $L(z)$ and the SW differential $dS^{\mathrm{Gaudin}}(y,z)=ydz$. The original argument [1] dealt with the $SU(2)$ case and implied that on the conformal side of the AGT correspondence the counterpart of the SW differential is played by the average of the energy-momentum tensor, and this latter shows up a pole behaviour which is rather associated with the Gaudin model. This argument was refined later by associating the SW differential with an insertion of the surface operator [6, 9, 11] or with the matrix model resolvent [7].

If considering the case of higher rank group $SU(N)$, which on the gauge theory side is associated with the Heisenberg chain (on N sites), one has to take into account that on the conformal side the AGT conjecture in this case deal with a four-point conformal block of the W_N-algebra [5], however not an arbitrary one but that restricted with special conditions imposed onto two of the four external operators (states) of the block. This

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means that there are two arbitrary operators parameterized by $N-1$ parameters each and two other operators parameterized by only one parameter each. In integrable terms this means that one should expect for the associated integrable system, the reduced Gaudin model that it is described by two coadjoint orbits of the maximal dimensions inserted in two points, and by two coadjoint orbits of the minimal dimensions inserted in two other points. As we shall see, this is, indeed, the case.

In this letter we show that the change of variables $z = w$, $y = x/w$ relates the curves and SW differentials of the two integrable systems under discussion (the Heisenberg spin chain and the reduced Gaudin model). It means that with this change of variables the following relations hold true:

$$\begin{gathered}\Gamma^{\rm Gaudin}(y,z) = \Gamma^{\rm Heisen}(z,zy),\\ dS^{\rm Gaudin}(y,z) = dS^{\rm Heisen}(z,zy).\end{gathered} \tag{1}$$

This type of relations between spectral curves appeared in [22][2)].

Let a pair of (algebraically) integrable models be described by the spectral curves $\Gamma(\lambda,z)=0$, $\Gamma'(\lambda',z') = 0$ and the corresponding SW differentials $dS(\lambda,z)$, $dS'(\lambda',z')$. Then the models are called *spectrally dual* (this term is borrowed from [23]) at the classical level if there exists a change of variables

$$\lambda' = \lambda'(\lambda,z), \quad z' = z'(\lambda,z)$$

such that

$$\begin{gathered}\Gamma(\lambda,z) = \Gamma'[z'(\lambda,z),\lambda'(\lambda,z)]\\ \text{and}\\ dS(\lambda,z) \cong dS[\lambda'(\lambda,z),z'(\lambda,z)],\end{gathered} \tag{2}$$

where $\cong$ emphasizes that the SW differential for the integrable system is determined up to a full differential on the spectral curve.

The duality transformation acts by *bispectral involution* [24] which interchanges the roles of the eigenvalue-variable and spectral parameter.

A well-known simpler example (found by L. Faddeev and L. Takhtajan [25]) is the periodic Toda chain. It can be described by both the gl(N)-valued Lax matrix:

$$L^{\rm Toda}_{N\times N}(z) = \tag{3}$$
$$\begin{pmatrix} p_1 & e^{\frac12(q_2-q_1)} & 0 & & ze^{\frac12(q_1-q_N)}\\ e^{\frac12(q_2-q_1)} & p_2 & e^{\frac12(q_3-q_2)} & \dots & 0\\ 0 & e^{\frac12(q_3-q_2)} & p_3 & & 0\\ & & \dots & & \\ \frac1z e^{\frac12(q_1-q_N)} & 0 & 0 & & p_N\end{pmatrix}$$

and the GL(2)-valued transfer-matrix:

$$\begin{gathered}T^{\rm Toda}_{2\times 2}(\lambda) = L_N(\lambda)...L_1(\lambda),\\ L_i(\lambda) = \begin{pmatrix}\lambda - p_i & e^{q_i}\\ -e^{-q_i} & 0\end{pmatrix},\\ i=1,\dots,N.\end{gathered} \tag{4}$$

The spectral curves defined by these representations are related by the bispectral involution, i.e.

$$\det[\lambda - L^{\rm Toda}_{N\times N}(z)] = 0 \quad \text{and} \quad \det[z - T^{\rm Toda}_{2\times2}(\lambda)] = 0 \tag{5}$$

coincide. The SW differential is the same in both cases $dS = \lambda\frac{dz}{z}$. Therefore, *the periodic Toda chain is a self-dual model* [26].

The quantum version of the duality appears from the exact quasi-classical quantization of the spectral curves. Considering the SW differential as a symplectic 1-form [27] on $\mathbb{C}\times\mathbb{C}^*$-plane (y,z) yields a pair of canonical variables $(p(y,z),q(z))$ which brings the SW differential to $dS(y,z) = pdq$. Then there is a natural quantization of the spectral curve defined by the rule $(p,q)\to(\hbar\partial_q, q)$.

Let two integrable models be described by the Baxter equations

$$\hat\Gamma\Psi = 0 \quad \text{and} \quad \hat\Gamma'\Psi' = 0. \tag{6}$$

They are called *spectrally dual at the quantum level* if their Baxter equations coincide.

For the above mentioned models one has:

$$\hat\Gamma^{\rm Heisen}(z,\hbar z\partial_z)\Psi^{\rm Heisen}(z) = 0, \tag{7}$$

$$\hat\Gamma^{\rm Gaudin}(\hbar\partial_z, z)\Psi^{\rm Gaudin}(z) = 0 \tag{8}$$

with some choice of ordering. The wave functions can be written in terms of the quantum deformation of the SW differential on the spectral curve, i.e. $\Psi(z) = \exp\left[-\frac1\hbar\int^q dS(\hbar)\right]$, where $dS(\hbar) = p(q,\hbar)dq$ and $p(q,0) = p(q)|_\Gamma$. The monodromies of the wave function

2) We call the duality between Gaudin models described in [22] as AHH (Adams–Harnad–Hurtubise) duality. See the comment in the end of the paper.

around A- and B-cycles of Γ are given by the quantum deformed action type variables [4]:

$$\Psi(z+A_i)=\exp\left(-\frac{1}{\hbar}a_i^{\hbar}\right)\Psi(z),\quad a_i^{\hbar}=\oint\limits_{A_i}dS(\hbar),$$

$$\Psi(z+B_i)=\exp\left(-\frac{1}{\hbar}\frac{\partial\mathcal{F}_{\rm NS}}{\partial a_i^{\hbar}}\right)\Psi(z),\ \frac{\partial\mathcal{F}_{\rm NS}}{\partial a_i^{\hbar}}=\oint\limits_{B_i}dS(\hbar), \tag{9}$$

where $\mathcal{F}_{\rm NS}$ is the Nekrasov–Shatashvili limit [3] of the LMNS integral [16].

The AGT conjecture predicts the following relations:

$$a_i^{\hbar}(\Psi^{\rm Heisen})=a_i^{\hbar}(\Psi^{\rm Gaudin}),$$
$$\frac{\partial\mathcal{F}_{\rm NS}}{\partial a_i^{\hbar}}(\Psi^{\rm Heisen})=\frac{\partial\mathcal{F}_{\rm NS}}{\partial a_i^{\hbar}}(\Psi^{\rm Gaudin}), \tag{10}$$

which imply the quantum spectral duality.

In this paper we deal with the known quantum equation (7) for the XXX chain – the Baxter equation[3] [29]:

$$\left[{\rm tr}T(\hbar z\partial_z)-\frac{z}{1+q}K_+(\hbar z\partial_z)-\frac{q}{(1+q)z}K_-(\hbar z\partial_z)\right]\times$$
$$\times\,\Psi^{\rm Heisen}(z)=0. \tag{11}$$

We verify that (11) can be re-written as the quantum spectral curve of the Gaudin model (8). In this way we arrive to the quantum version of duality:

$$\Psi^{\rm Heisen}(z)=\Psi^{\rm Gaudin}(z). \tag{12}$$

Below we briefly describe the models and formulate the spectral duality. Some comments are given at the end. Most of details can be found in [30]. In that extended version we also describe the Poisson map between models.

1. N-site GL_2 Heisenberg (XXX) chain. It is classically defined by its spectral curve

$$\Gamma^{\rm Heisen}(w,x):\quad {\rm tr}T(x)-\frac{1}{1+q}wK^+(x)-$$
$$-\frac{q}{1+q}w^{-1}K^-(x)=0,\ K^{\pm}(x)=\prod_{i=1}^{N}(x-m_i^{\pm}) \tag{13}$$

and SW differential

$$dS^{\rm Heisen}(w,x)=x\frac{dw}{w}. \tag{14}$$

[3] It arises as an equation for the Baxter Q-operator eigenvalues in the Quantum Inverse Scattering Method [28]. Originally, it was written in difference (Fourier-dual) form.

$T(x)$ in (13) is GL_2-valued transfer-matrix:

$$T(x)=VL_N(x)\dots L_1(x),\quad L_i(x)=x-x_i+S^i,$$
$$i=1\dots N,\quad V=\begin{pmatrix}1 & -\dfrac{q}{(1+q)^2}\\ 1 & 0\end{pmatrix},$$
$$S^i\in{\rm sl}_2:\ {\rm Spec}(S^i)=(K_i,-K_i),\quad m_i^{\pm}=x_i\pm K_i. \tag{15}$$

Function

$${\rm tr}T(x)=x^N+\sum_{i=1}^{N}x^{i-1}H_i^{\rm Heisen} \tag{16}$$

provides commuting integrals of motion. The spectral curve (13) follows directly from its definition $\det[w-T(x)]=0$ with the change $w\to w\frac{1}{1+q}K^+(x)$.

2. Special (reduced) gl_N Gaudin model on $\mathbb{CP}^1\backslash\{0,1,q,\infty\}$. It is described by the spectral curve

$$\Gamma^{\rm Gaudin}(y,z):\quad \det[y-L(z)]=0,$$
$$L(z)=\frac{A^0}{z}+\frac{A^1}{z-1}+\frac{A^q}{z-q}\in{\rm gl}_N \tag{17}$$

with additional conditions including the reduction constraints[4]

$$A^0+A^1+A^q+A^{\infty}=0,$$
$$A^{\infty}\equiv\Upsilon={\rm diag}(\upsilon_1,...,\upsilon_N),\ {\rm Spec}(A^0)=(\mu_1,...,\mu_N),$$
$$A^1=\xi^1\times\eta^1,\quad A^q=\xi^q\times\eta^q, \tag{18}$$

i.e. A^1 and A^q are gl_N matrices of rank 1 (this type of configuration was already discussed [31, 10]). Using specification (18) the spectral curve can be find explicitly using the following simple fact: For any given invertible matrix $G\in{\rm Mat}(N)$ and a pair of N-dimensional vectors ξ and η

$$\det(G+\xi\eta^{\rm T})=(1+\eta^{\rm T}G^{-1}\xi)\det G$$

and

$$(G+\xi\eta^{\rm T})^{-1}=G^{-1}-\frac{1}{1+\eta^{\rm T}G^{-1}\xi}G^{-1}\xi\eta^{\rm T}G^{-1}.$$

Then for $\Gamma^{\rm Gaudin}(y,z)$ we have

$$[\eta^1(zy+\Upsilon)^{-1}\xi^1+q\eta^q(zy+\Upsilon)^{-1}\xi^q+q+1]\prod_{i=1}^{N}(zy+\upsilon_i)$$
$$=z\prod_{i=1}^{N}(zy+\upsilon_i)+z^{-1}q\prod_{i=1}^{N}(zy-\mu_i) \tag{19}$$

[4] One should also fix the action of the Cartan subgroup. We do not discuss it here since it does not effect the curve.

or

$$\prod_{i=1}^{N}(zy+\upsilon_i)+\sum_{k=1}^{N}\frac{\eta_k^1\xi_k^1+q\eta_k^q\xi_k^q}{q+1}\prod_{i\neq k}^{N}(zy+\upsilon_i)=$$
$$=\frac{z}{q+1}\prod_{i=1}^{N}(zy+\upsilon_i)+z^{-1}\frac{q}{q+1}\prod_{i=1}^{N}(zy-\mu_i). \quad (20)$$

The SW differential is

$$dS^{\text{Gaudin}}(y,z)=ydz. \quad (21)$$

The classical spectral duality. First, notice that the both models (as classical mechanical systems) describe dynamics of $N-1$ degrees of freedom and depend on $2N+1$ parameters.

Indeed, the dynamical variables of the off-shell Gaudin model (18) are $A^{0,1,q,\infty}$. Fixing the Casimir functions restricts $A^{0,1,q,\infty}$ to the coadjoint orbits of maximum dimensions (N^2-N) at $z=0,\ \infty$ and of minimal dimensions $(2N-2)$ at $z=1,\ q$. Then the reduction by the coadjoint action of GL_N gives the following dimension of the phase space:

$$2(N^2-N)+2(2N-2)-2(N^2-1)=2(N-1). \quad (22)$$

The number of parameters is $2N+3$: $\{\upsilon_1,...,\upsilon_N,\mu_1,...,\mu_N,\mathrm{tr}A^1,\mathrm{tr}A^q,q\}$. Two of them, $(\mathrm{tr}A^0,\mathrm{tr}A^\infty)$ can be eliminated from the spectral curve by the shift of y. Therefore, the number of independent parameters is $2N+1$.

For the Heisenberg chain, one initially has N sl_2-valued variables S^i with the Casimir functions fixed at each site: $\frac{1}{2}\mathrm{tr}\left(S^i\right)^2=K_i^2$. The reduction by $\mathrm{Stab}[V(q)]\cong\mathrm{Cartan}(\mathrm{GL}_2)$ fixes two independent variables. Therefore, for the dimension of the phase space one has

$$3N-N-2=2(N-1) \quad (23)$$

and there are $2N+1$ parameters $\{x_1,...,x_N,K_1,...,K_N,q\}$.

The duality between models is described in the following way.

The N-site GL_2 Heisenberg XXX chain defined by (13)–(16) and the gl_N Gaudin model (17)–(21) are spectrally dual at the classical level

$$\begin{aligned}\Gamma^{\text{Gaudin}}(y,z)&=\Gamma^{\text{Heisen}}(w,x),\\ dS^{\text{Gaudin}}(y,z)&=dS^{\text{Heisen}}(w,x)\end{aligned} \quad (24)$$

with the following change of variables

$$z=w,\quad y=x/w, \quad (25)$$

identification of parameters

$$m_i^+=-\upsilon_i,\quad m_i^-=\mu_i,\quad 1\le i\le N, \quad (26)$$

and relation between generating functions of the Hamiltonians:

$$\mathrm{tr}T^{\text{Heisen}}(y)=\det(y+\Upsilon)\times$$
$$\times\left[1+\frac{1}{1+q}\eta^1(y+\Upsilon)^{-1}\xi^1+\frac{q}{1+q}\eta^q(y+\Upsilon)^{-1}\xi^q\right]. \quad (27)$$

The proof of the statements follows from the comparison between (13), (14), and (19)–(21). In particular,

$$H_N^{\text{Heisen}}=\frac{1}{1+q}\mathrm{tr}A^1+\frac{q}{1+q}\mathrm{tr}A^q+\sum_{k=1}^{N}\upsilon_k. \quad (28)$$

In the abstract the equivalence between the models was announced. The equivalence should be understood as (24). Indeed, the spectral curves describe the models in terms of the separated variables while the SW differentials encode the symplectic structures in terms of these variables.

The quantum spectral duality. The quantization of the XXX chain spectral curve (19) with the SW differential (14) means that x should be simply replaced by $\hbar w\partial_w$. Then one gets the Baxter equation:

$$\left[\mathrm{tr}\,T(\hbar w\partial_w)-\frac{w}{1+q}K_+(\hbar w\partial_w)-\right.$$
$$\left.-\frac{q}{(1+q)w}K_-(\hbar w\partial_w)\right]\Psi^{\text{Heisen}}(w)=0. \quad (29)$$

Equivalently, for the Gaudin spectral curve (20) the quantization is given by the replacement $y\to\hbar\partial_z$:

$$\left[\prod_{i=1}^{N}(z\hbar\partial_z+\upsilon_i)+\sum_{k=1}^{N}\frac{\eta_k^1\xi_k^1+q\eta_k^q\xi_k^q}{q+1}\prod_{i\neq k}^{N}(z\hbar\partial_z+\upsilon_i)-\right.$$
$$\left.-\frac{z}{q+1}\prod_{i=1}^{N}(z\hbar\partial_z+\upsilon_i)-z^{-1}\frac{q}{q+1}\prod_{i=1}^{N}(z\hbar\partial_z-\mu_i)\right]\times$$
$$\times\Psi^{\text{Gaudin}}(z)=0. \quad (30)$$

Let us summarize the quantum version of the spectral duality. The N-site GL_2 Heisenberg XXX chain defined by (13)–(15), (29) and the gl_N Gaudin model (17)–(18), (30) are spectrally dual at the quantum level with the following relation between the wave functions:

$$\psi_{\text{XXX}}(z)=\psi_{\text{Gaudin}}(z)e^{\frac{1}{N\hbar}\int^z b_\hbar\,dz}, \quad (31)$$

where $b_\hbar$ is given below (37), i.e. the Baxter equation of the Heisenberg chain can be rewritten as the quantization of the Gaudin model spectral curve.

Obviously, the differential operators in the brackets of (29) and (30) can be identified in the same way as the classical spectral curves did.

Let us comment the obtained results and their relations to known constructions.

- *AHH duality.* In [22] (see also [32]) the authors considered the Gaudin model with M marked points and the Lax matrix defined as follows:

$$L^G_{AHH}(z) = Y + \sum_{c=1}^{M} \frac{A^c}{z - z_c}, \quad Y = \text{diag}\,(y_1, ..., y_N), \quad A^c \in \text{gl}_N. \tag{32}$$

The later differs from ours. The difference is significant since $Y \neq 0$ leads to the second order pole at ∞ for $L^G_{AHH}(z)dz$. The phase space is also different. It is a direct product of the coadjoint orbits (equipped with a natural Poisson–Lie structure) factorized by the stabilizer of Y: $\mathcal{O}^1 \times \cdots \times \mathcal{O}^M // \text{Stab}(Y)$.

In the case when all A^c are of rank 1 the dual Lax matrix is the gl_M-valued function with $\tilde{Y} = \text{diag}(z_1, ..., z_M)$ and N marked points at $y_1, ..., y_N$:

$$\tilde{L}^G_{AHH}(z) = \tilde{Y} + \sum_{c=1}^{N} \frac{\tilde{A}^c}{z - y_c}, \quad \tilde{Y} = \text{diag}(z_1, ..., z_M), \quad \tilde{A}^c \in \text{gl}_M. \tag{33}$$

The duality implies the following relation between the spectral curves:

$$\det(\tilde{Y} - z)\det[L^G_{AHH}(z) - \lambda] = \det(Y - \lambda)\det[\tilde{L}^G_{AHH}(\lambda) - z]. \tag{34}$$

The dimensions of the phase spaces of both models equal $2(N-1)(M-1)$ and the number of parameters is $2(N+M)-3$.

- Sometimes sl_N description of the Gaudin model is more convenient than the gl_N one. The transformation of the spectral curve from gl_N to sl_N is given by the simple shift:

$$y \to y' = y - \frac{1}{N}\text{tr}L(z) = y - \frac{1}{zN}\left(-\text{tr}\Upsilon + \frac{\text{tr}A^1}{z-1} + q\frac{\text{tr}A^q}{z-q}\right). \tag{35}$$

In this case the change of variables (25) is modified[5)]:

$$z = w, \quad y' = \frac{x - R(z)}{w}, \quad R(z) = \frac{1}{N}\left(-\text{tr}\Upsilon + \frac{\text{tr}A^1}{z-1} + q\frac{\text{tr}A^q}{z-q}\right). \tag{36}$$

The equality of the wave functions (12) acquires the predictable multiple:

$$\Psi^{\text{Heisen}}(z) = \Psi^{\text{Gaudin}}(z) e^{\frac{1}{N\hbar}\int^z b_\hbar(z)dz},$$

$$b_\hbar(z) = \frac{1+q}{(z-1)(z-q)} \times \left(H^{\text{Heisen}}_N + \frac{z\sum_{k=1}^N m_k^+}{1+q} + \frac{q\sum_{k=1}^N m_k^-}{(1+q)z}\right) - \hbar\frac{N(N-1)}{2z}. \tag{37}$$

- It should be mentioned that we do not impose any boundary conditions which provide a valuable quantum problem, i.e. we do not specify wave functions explicitly. To compare the quantum problems one needs a construction of the Poisson (and then quantum) map between the phase spaces (Hilbert spaces) of the two models. At classical level this map is given in [30].

Alternatively, one can specify the spaces of solutions initially and then verify their identification through the duality transformation. This is the recipe of [33] where the authors considered duality between the XXX chain and trigonometric Gaudin model at the quantum level in terms of the corresponding Bethe vectors. Their description is based on the linear (Lie algebra) commutation relations, while the Gaudin model under consideration here is the rational (although reduced) one. The Poisson structure (which is discussed at the classical level) is quadratic. The precise connection between theses two cases deserves further elucidation.

- Besides the approach proposed here, a quantization of the Gaudin model is known from [34] and [35]. We hope to shed light on relations between the quantizations in further publications.

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5) In this form the change of variables was found in [12] for sl_2 case.

- As is well known, the sl_2 reduced Gaudin model with the configuration discussed above can be written in different elliptic forms [36–39] with q be a function of the modular parameter. Therefore, one can expect some elliptic parametrization for the sl_N case as well. Another remark is that the Gaudin-type models are related to many-body systems via modification procedure [40]. We plan to return to these issues in further publications.

- At last, let us mention possible generalizations of the correspondence proposed in this letter. First of all, one can naturally consider multi-point conformal blocks. This provides one with the multi-point Gaudin model. At the same time, the AGT predicts in this case on the other side of the correspondence the theory with gauge group being a product of a few gauge factors. This latter is naturally embedded into the spin magnets with higher rank group [26]. Thus correspondence between $GL(p)$-magnets and multi-point Gaudin models discussed in [30].

  Another interesting generalization is induced by the five-dimensional AGT [41] which implies a correspondence between the XXZ magnets (see [42]) and a Gaudin-like model with relativistic (difference) dynamics. This latter would emerge, since on the conformal side one deal in this case with the q-Virasoro conformal block which implies a difference Schrödinger equation for the block with insertion of the degenerate field. An extension to six dimensions (elliptic extension of the differential operator in the Schrödinger equation versus XYZ magnet) is also extremely interesting to construct.

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