

# Luttinger liquid with one impurity: equivalent field theory and duality

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We derive by functional integral method 1-dimensional theory equivalent to Luttinger liquid with one impurity. It is shown that the single quantity has to be calculated for the formulation the effective field theory is the electron density jump at the impurity. We show that duality which relates models with electron-electron attraction and repulsion, and simultaneous exchange of transition and reflection coefficients, is an exact property of the theory.

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The theory of 1-dimensional interacting fermions is developing for more than 50 years. This theory is interesting as nowadays the corresponding systems can be accessed experimentally. One-dimensional interacting electrons are well understood: this problem can be solved exactly either in terms of left and or right moving electrons or by method of bosonization. Situation changes if impurity back scattering is possible. In the simplest case of only one point-like impurity introduced into the system the surprising results were obtained [1]: for electrons with repulsion the static conductivity of the channel vanishes. In the above paper Hamiltonian was based on the physical considerations which are valid only in the first order in scattering on impurity; the problem was solved by method of bosonization. Another approach using renorm-group technique was developed in [2]. In this approach the oscillating (with period  $1/p_F$ ) effective long range  $e-e$  potential appears. This potential is a mixture of original  $e-e$  interaction and interaction with impurity. Both approaches gave coinciding results in the first order of perturbation theory. Zero conductance was interpreted as a result of electron scattering on Friedel oscillations.

In this letter we propose yet another approach to the problem. We reduce the theory of one-dimensional and one-component interacting electrons in a system with impurity to effective zero-dimensional field theory (i.e. to some quantum mechanics with Hamiltonian not local in time). The effective theory describes dynamics of the phase of the electron Green function at the point where impurity is located. We will assume that the size of impurity is comparable with  $p_F^{-1}$  while the radius of  $e-e$  interaction is much larger. Only electrons with momenta  $\pm p_F + p$  (left and right movers) where  $|p| \ll p_F$ , enter the problem. The impurity

can be characterized by reflection ( $\mathcal{R}$ ) and transition ( $\mathcal{K}$ ) coefficients which are the constants calculated at  $p = 0$ .

The problem has a number of properties already seen in the first orders of perturbation theory [1]. In particular, it reveals so-called *duality* which implies that if conductivity  $\mathcal{G}$  is known for the theory with attraction between electrons, then one can obtain it in the theory with repulsion, simply exchanging reflection and transition coefficients. Breakdown of duality was reported in [3]. We will prove in this letter that duality is an exact property of the theory and, in particular, it is valid for the exact reflection and transition coefficient.

It is convenient to understand  $e-e$  interaction according to Hubbard [4], i.e. reduce it to the problem of non-interacting electrons in the arbitrary external field  $U(x, t)$ . The interaction is invoked as a result of integration in all possible  $U(x, t)$ . Integrating out fermion degrees of freedom one can reduce the problem to calculation of the current in the external field and fermion determinant which is an effective action for the external field  $U(x, t)$ . The determinant is a sum of fermion loops in the external field. Both quantities can be expressed in terms of the Green function in presence of impurity.

External slowly varying field  $U(x, t)$  does not change the direction of electron movement, so left and right electrons do not mix everywhere except the point  $x = 0$  where impurity is located. The impurity cannot be modeled by the term  $\Psi_R^+(0, t)\Psi_L(0, t)\delta(x)$  in the Hamiltonian of the system. To account for impurity correctly one has to return to the second order Schrödinger equation, calculate reflection and transition coefficients and separate once more large momentum  $p_F$  substituting these (depending on momentum) coefficients by their

values at  $p = p_F$ . Outside the impurity Green function obeys first order Dirac equation:

$$[i\partial_t \pm i\partial_x - U(x, t)] G_{R,L,\dots}(x, x') = i\delta^{(2)}(x - x') \quad (1)$$

(we work in the system of units where  $e, \hbar, v_F = 1$ ). Impurity should be understood as specific boundary condition at  $x = 0$ . Green function has to obey Feynman boundary conditions at  $t \rightarrow \pm\infty$ .

We build a full set of functions obeying homogeneous Eq. (1) and boundary conditions which describe scattering on impurity:

$$\begin{aligned} \psi_\varepsilon^{(1)}(x, t) &= \begin{pmatrix} [\mathcal{K}^* \Theta(-x) + \Theta(x)] e^{-i\varepsilon(t-x)} e^{i\gamma_R(x,t)} \\ \mathcal{R}^* \Theta(x) e^{i\gamma_L(x,t) + i\alpha(t+x)} e^{-i\varepsilon(t+x)} \end{pmatrix}, \\ \psi_\varepsilon^{(2)}(x, t) &= \begin{pmatrix} \mathcal{R}^* e^{-i\varepsilon(t-x)} \Theta(-x) e^{i\gamma_R(x,t) - i\alpha(t-x)} \\ [\Theta(-x) + \mathcal{K}^* \Theta(x)] e^{-i\varepsilon(t+x)} e^{i\gamma_L(x,t)} \end{pmatrix}, \\ \psi_{-\varepsilon}^{(1)}(x, t) &= \begin{pmatrix} [\Theta(-x) + \Theta(x)\mathcal{K}] e^{i\varepsilon(t-x)} e^{i\gamma_R(x,t)} \\ \mathcal{R}\Theta(-x) e^{i\varepsilon(t+x)} e^{i\gamma_L(x,t)} e^{+i\alpha(t+x)} \end{pmatrix}, \\ \psi_{-\varepsilon}^{(2)}(x, t) &= \begin{pmatrix} \Theta(x)\mathcal{R} e^{i\varepsilon(t-x)} e^{i\gamma_R(x,t)} e^{-i\alpha(t-x)} \\ [\mathcal{K}\Theta(-x) + \Theta(x)] e^{i\varepsilon(t+x)} e^{i\gamma_L(x,t)} \end{pmatrix}. \quad (2) \end{aligned}$$

Here,  $\varepsilon$  is energy of electron at  $|t| \rightarrow \infty$ , indices label direction of movement of incident electron ("1" from  $x = -\infty$ , "2" from  $x = \infty$ ) and  $\gamma_{R,L}$  are phases:

$$\begin{aligned} \gamma_{R,L}(x, t) &= -\int d^2 x' G_{R,L}^{(0)}(x, t, x', t') U(x', t'), \\ \alpha(t) &= \gamma_R(0, t) - \gamma_L(0, t). \quad (3) \end{aligned}$$

We construct the Green function from full set of states (2) as follows:

$$\begin{aligned} G_{\alpha\beta}(xt, x't') &= \sum_{i,k=1}^2 \int \frac{d\varepsilon d\varepsilon'}{(2\pi)^2} S_{ik}(\varepsilon, \varepsilon') \psi_{\alpha\varepsilon}^{(i)}(xt) \bar{\psi}_{\beta\varepsilon'}^{(k)}(x't') \times \\ &\times [\theta(t-t')\theta(\varepsilon)\theta(\varepsilon') - \theta(t'-t)\theta(-\varepsilon)\theta(-\varepsilon')]. \quad (4) \end{aligned}$$

Here,  $\bar{\psi}$  are conjugate solutions which are obtained from (2) by complex conjugation, the exponents as if  $\gamma_{R,L}$  would be real and replacement  $x \rightarrow -x$  at pre-exponential factors (without complex conjugation).

It is easy to check that Green function (4) obeys equation (1) (everywhere except  $x = x'$ ), matching condition at  $x = 0$  and Feynman boundary conditions. In order to get  $\delta$ -function in r.h.s. one has to require

$$\begin{aligned} \sum_{i,k=1}^2 \int \frac{d\varepsilon d\varepsilon'}{(2\pi)^2} S_{ik}(\varepsilon, \varepsilon') \psi_{\alpha\varepsilon}^{(i)}(xt) \bar{\psi}_{\beta\varepsilon'}^{(k)}(x't) \theta(\varepsilon\varepsilon') &= \\ = \delta_{\alpha\beta} \delta(x - x'). \quad (5) \end{aligned}$$

This equation is equivalent to the following Wiener-Hopf system:

$$\begin{aligned} \mathcal{K}[S_{11}(\tau, -\varepsilon_2)]_+ + \mathcal{R}[e^{-i\alpha(\tau)}[S_{21}(\tau, -\varepsilon_2)]_+]_+ &= e^{i\varepsilon_2\tau}, \\ \mathcal{K}[S_{21}(\tau, -\varepsilon_2)]_+ + \mathcal{R}[e^{i\alpha(\tau)}[S_{11}(\tau, -\varepsilon_2)]_+]_+ &= 0 \quad (6) \end{aligned}$$

( $S(\tau, \varepsilon_2)$  is Fourier transform of  $S(\varepsilon_1, \varepsilon_2)$  in the first argument). We denote here by  $[F(\tau)]_+$  the part of the function  $F(\tau)$  which is analytical in the upper semi-plane of  $\tau$ . One can easily iterate (6) at small  $|\mathcal{R}|^2$

$$\begin{aligned} [S_{21}]_+ &= -\frac{\mathcal{R}}{\mathcal{K}^2} [e^{i\alpha+i\varepsilon_2\tau}]_+ - \\ -\frac{\mathcal{R}^3}{\mathcal{K}^4} [e^{i\alpha} [e^{-i\alpha} [e^{i\alpha+i\varepsilon_2\tau}]_+]_+]_+ &- \dots \quad (7) \end{aligned}$$

In the opposite limiting case  $\mathcal{K} \ll 1$  it is convenient to proceed to new functions  $\sigma_{ik}(\tau)$

$$\begin{aligned} [S_{11}(\tau)]_\pm &= e^{-i\alpha_\pm(\tau)} [\sigma_{21}(\tau)]_\pm, \\ [S_{21}(\tau)]_\pm &= e^{i\alpha_\pm(\tau)} [\sigma_{11}(\tau)]_\pm. \quad (8) \end{aligned}$$

Then the system (6) transforms to:

$$\begin{aligned} \mathcal{R}[\sigma_{11}(\tau, -\varepsilon_2)]_+ + \mathcal{K}[e^{-i\tilde{\alpha}(\tau)}[\sigma_{21}(\tau, -\varepsilon_2)]_+]_+ &= \\ = e^{i\varepsilon_2\tau + i\alpha_-(\tau)}, \\ \mathcal{R}[\sigma_{21}(\tau, -\varepsilon_2)]_+ + \mathcal{K}[e^{i\tilde{\alpha}(\tau)}[\sigma_{11}(\tau, -\varepsilon_2)]_+]_+ &= 0. \quad (9) \end{aligned}$$

Here  $\alpha_\pm(\tau) = [\alpha(\tau)]_\pm$  and  $\tilde{\alpha}(\tau) = \alpha_+(\tau) - \alpha_-(\tau)$ . The system (9) is *dual* to the system (6). From this system it is easy to construct expansion at small  $\mathcal{K}$  which is analogous to Eq. (7).

The current in the external field  $j(x, t) = \rho_R(x, t) - \rho_L(x, t)$  is the limit of the Green function at the coinciding arguments. It is well known that this limit is ill-defined due to the ultraviolet problems. The correct definition which supports the gauge invariance (and hence conservation of the current) can be derived by the so-called symmetric splitting of the arguments [5]. The asymptotic of  $S_{ik}$  at large  $\varepsilon_{1,2}$  is given by:

$$S_{ik}^{(as)}(\varepsilon_1 - \varepsilon_2) = \begin{pmatrix} 2\pi\mathcal{K}^*\delta(\varepsilon_1 - \varepsilon_2) & \mathcal{R}^*\phi_-(\varepsilon_2 - \varepsilon_1) \\ \mathcal{R}^*\phi_+(\varepsilon_2 - \varepsilon_1) & 2\pi\mathcal{K}^*\delta(\varepsilon_1 - \varepsilon_2) \end{pmatrix}, \quad (10)$$

where

$$\phi_\pm(\varepsilon) = \int dz \exp[i\varepsilon z \pm i\alpha(z)].$$

We divide the current into two parts  $j = j^{(uv)} + j^{(reg)}$ : ultraviolet part  $j^{(uv)}$  should be calculated with splitted arguments of the Green function and using  $S_{ik}^{(as)}$  and regular part  $j^{(reg)}$  is obtained by substituting  $S_{ik}$  by the ultraviolet regular part  $S_{ik} - S_{ik}^{(as)}$  and at coinciding arguments. The full current is conserved and reproduces one-dimensional Adler anomaly [5].

Direct calculation gives the following expression for ultraviolet part of  $R, L$  densities:

$$[\rho_{R,L}(x,t)]^{uv} = \frac{|\mathcal{R}|^2}{(2\pi)^2} \int d\tau \left[ \frac{e^{-i\alpha(\tau)+i\alpha(x_{\mp})}}{(\tau - x_{\mp} \mp i\delta)^2} \theta(-x) + \frac{e^{i\alpha(\tau)-i\alpha(x_{\mp})}}{(\tau - x_{\mp} \pm i\delta)^2} \theta(x) \right], \quad (11)$$

(here  $x_{\pm} = t \pm x$ ). Current is  $j = \rho_R - \rho_L$  and charge density  $\rho = \rho_R + \rho_L$ . Regular part of the current at small reflection coefficient starts from the term  $|\mathcal{R}|^4$ .

Conservation of the current and Adler anomaly

$$\partial_t \rho + \partial_x j = 0, \quad \partial_t j + \partial_x \rho = -\frac{\partial_x U}{\pi} + \mathfrak{D}(t)\delta(x) \quad (12)$$

determines the current and charge density everywhere except point of impurity. Both quantities can be expressed in terms of charge jump

$$\mathfrak{D}(t) = \rho(x=+0, t) - \rho(x=-0, t)$$

(current  $j$  should be continuous) at this point as follows:

$$\begin{pmatrix} \rho[U] \\ j[U] \end{pmatrix} = \begin{pmatrix} ik \\ i\omega \end{pmatrix} \frac{ikU(k, \omega)/\pi + \mathfrak{D}[\alpha](\omega)}{\omega^2 - k^2 + i\delta}. \quad (13)$$

The first term is a ballistic current. The second term describes the effect of impurity. It depends only on  $\alpha(t)$ .

According to general formula:

$$\log \mathfrak{D}et[U] = -i \int_0^1 d\lambda \int \frac{d^2 k}{(2\pi)^2} U(-k, -\omega) \rho[\lambda U](k, \omega) \quad (14)$$

loop corrections consists of two parts. The sum of the first (ballistic) part  $\log \mathfrak{D}et_{\text{ball}}[U]$  and the action appearing as a result of Hubbard trick [4]

$$S_H[U] = \int \frac{d^2 k}{(2\pi)^2} \frac{U(k, \omega)U(-k, -\omega)}{2V(k)} \quad (15)$$

( $V(k)$  is Fourier transform of the  $e-e$  potential) produces an effective action of the pure Luttinger model [6]:

$$S_{\text{eff}}[U] = \frac{1}{2} \int \frac{d\omega dk}{(2\pi)^2} \frac{U(k, \omega)U(-k, -\omega)}{V(k)} \frac{\omega^2 - k^2 v_c^2(k)}{\omega^2 - k^2 + i\delta}. \quad (16)$$

Here  $v_c = \sqrt{1 + V(k)/\pi}$  is renormalized Fermi speed. In fact, Eq.(16) is Dzyaloshinskii-Larkin theorem [7] formulated in terms of functional integral.

The second part is an effect of impurity

$$\log \mathfrak{D}et_{\text{imp}} = -i \int_0^1 d\lambda \int \frac{d^2 k}{(2\pi)^2} \frac{ikU(-k, -\omega)}{\omega^2 - k^2 + i\delta} \mathfrak{D}[\lambda\alpha](\omega). \quad (17)$$

To calculate the Feynman response of the system to the infinitesimal electric potential  $\varphi(x)$  (which is related to the conductivity of the system) we have to shift  $U \rightarrow U + \varphi$  and expand up to the linear order:

$$j(x, t) = \frac{1}{Z} \int \mathcal{D}U e^{iS_H[U]} \int d^2 x \varphi(x', t') \times \frac{\delta}{\delta U(x', t')} \{j[U](x, t) \mathfrak{D}et_{\text{imp}}[U] \mathfrak{D}et_{\text{bal}}[U]\}. \quad (18)$$

Integrating in functional integral in  $U$  by parts we arrive at

$$j(x, t) = \frac{1}{Z} \int \mathcal{D}U j[U](x, t) \mathfrak{D}et[\alpha]_{\text{imp}} e^{iS_{\text{eff}}[U]} \times \int d^2 x' [-i\varphi(x', t')] \int \frac{d^2 k}{(2\pi)^2} \frac{U(k, \omega)}{V(k)} e^{ikx' - i\omega t'}. \quad (19)$$

It can be seen from (13) that essential part of the integrand in (19) depends only on the one-dimensional phase  $\alpha$  and not on the complete function 2-dimensional potential  $U(x, t)$ . Let us proceed to this variable by means of Faddeev-Popov method [8] introducing unity into functional integral (19):

$$1 = \int \mathcal{D}\alpha \delta \left\{ \alpha + \int \frac{dk}{2\pi} U(\omega, k) \frac{2ik}{\omega^2 - k^2 + i\delta} \right\}$$

and calculating integral in  $U(x, t)$  first (note that first factor in (17) is precisely to  $\alpha(-\omega)/2$ ). Finally we arrive at the following expression for the part of the current depending on impurity:

$$\delta j(\omega, k) = \frac{2\omega|\omega|}{\omega^2 - v_c^2 k^2 + i\delta} \frac{v_c(\omega)}{\pi} |\mathcal{R}_{\omega}|^2 \times \int \frac{dq}{2\pi} \frac{E(q, \omega)}{\omega^2 - v_c^2 q^2 + i\delta}. \quad (20)$$

Here  $E = -\partial_x \varphi$  is electric field and  $|\mathcal{R}_{\omega}|^2$  is effective reflection coefficient determined by the averaged charge jump:

$$(2\pi)\delta(\omega - \omega') |\mathcal{R}_{\omega}|^2 = \frac{i\pi}{v_c(\omega)|\omega|W(\omega)} \langle \langle \mathfrak{D}[\alpha](\omega)\alpha(-\omega') \rangle \rangle. \quad (21)$$

Here average is understood as a functional integral in phase  $\alpha$

$$\langle \langle \dots \rangle \rangle = \frac{1}{Z} \int \mathcal{D}\alpha \times \dots \times \exp \left[ -\frac{1}{2} \int \frac{d\omega}{2\pi} \frac{\alpha(-\omega)\alpha(\omega)}{W(\omega)} \right] \mathfrak{D}et_{\text{imp}} \quad (22)$$

and  $W$  is the effective 1-dimensional potential:

$$W(\omega) = i \int \frac{dk}{2\pi} \frac{4k^2 V(k)}{(\omega^2 - k^2 + i\delta)[\omega^2 - v_c^2(k)k^2 + i\delta]}. \quad (23)$$

For the point-like interaction  $V = V_0\delta(x)$  it is equal to:

$$W(\omega) = \frac{2\pi}{|\omega|} \left[ \frac{1}{v_c} - 1 \right]. \quad (24)$$

In order to calculate the conductance  $\mathfrak{G}$  one has to restore retarded response from Feynman one and proceed to the limit of small  $\omega$ . As a result for the weak dispersion case (i.e.  $k\partial_k v_c(k)/v_c(k) \ll 1$  at  $k \rightarrow 0$ ) one has

$$\mathfrak{G}(\omega) = \frac{1}{2\pi v_c(\omega)} (1 - |\mathcal{R}_\omega|^2). \quad (25)$$

To summarize: effect of impurity in the Luttinger model can be calculated in some 1-dimensional effective field theory Eq. (22) (non-local quantum mechanics) describing dynamics of the phase  $\alpha$  of the electron Green function. It is determined by the average (21) of the charge jump  $\mathfrak{D}[\alpha]$  at the point of impurity. Loop corrections can be also expressed in terms of  $\mathfrak{D}[\alpha]$  by means of Eq. (17).

It is easy to construct the perturbation theory for  $|\mathcal{R}_\omega|^2$  in reflection coefficient. From (24) it is obvious that such expansion has sense only at  $v_c < 1$  ( $e-e$  attraction). In the lowest order the charge jump is obtained from the ultraviolet part of the current (11). Calculating Gaussian integral (21) for the point-like interaction we arrive at

$$\begin{aligned} |\mathcal{R}_\omega|^2 &= \frac{|\mathcal{R}|^2}{2\pi|\omega|} \int d\xi \frac{1 - \cos(\omega\xi)}{\xi^2} \Xi(\xi) = \\ &= \frac{2}{v_c} \Gamma(-1 - 2\nu) \frac{\sin \pi\nu}{\pi} |\mathcal{R}|^2 \left( \frac{|\omega|}{M} \right)^{2\nu} \end{aligned} \quad (26)$$

with  $\nu = 1/v_c - 1$ . Here

$$\begin{aligned} \Xi(\tau) &= \exp \left[ -\frac{1}{2} \int \frac{d\omega}{2\pi} W(\omega) |1 - e^{i\omega\tau}|^2 \right] = \\ &= \begin{cases} 1/(M|\tau|)^{2\nu} & |\tau| \gg 1/M, \\ 1 & |\tau| \ll 1/M \end{cases} \end{aligned} \quad (27)$$

and  $M$  is an ultraviolet cut-off which is determined by the radius of  $e-e$  interaction (or thickness of the channel). Expression (26) is valid only at  $2\nu < 1$ . At  $2\nu > 1$  the main contribution to (26) comes from  $\xi \leq M^{-1}$  and

$$|\mathcal{R}_\omega|^2 \cong |\mathcal{R}|^2 \frac{|\omega|}{M}. \quad (28)$$

In other words, for sufficiently large interaction expansion of  $\mathfrak{G}(\omega)$  at small  $\omega$  is regular and is determined by  $e-e$  potential at small distances.

Let us proceed with repulsive potential  $v_c > 1$ . In this case quadratic form in (21) is negative and expansion in  $\mathcal{R}$  becomes an expansion in inverse powers of

$\omega$  which blows up at small frequencies. For this reason one has to consider expansion in  $\mathcal{K}$  which is produced by dual Eq. (9). We proceed in the functional integral to the new variable  $\alpha(\omega) = -\text{sign}(\omega)\tilde{\alpha}(\omega)$  and quadratic form changes its sign. At small  $\mathcal{K}$  expansion of the charge jump contains a term independent on  $\mathcal{K}$  and linear in  $\alpha$

$$\mathfrak{D}[\tilde{\alpha}] = i\omega \frac{\tilde{\alpha}(\omega)}{\pi} + \text{sign}(\omega) \tilde{\mathfrak{D}}[\tilde{\alpha}](\omega). \quad (29)$$

“Dual” charge jump  $\tilde{\mathfrak{D}}[\tilde{\alpha}](\omega)$  depends on  $\tilde{\alpha}$  and has a regular expansion in transition coefficient starting from terms  $|\mathcal{K}|^2$ .

The first term in (29) plays two roles. First, it cancels completely the ballistic current in  $j[U](x, t)$ . Second, by means of (17) it leads to quadratic in  $\tilde{\alpha}$  correction to  $\log \mathfrak{D}et_{\text{imp}}$ . We will include this correction to the quadratic form which becomes:

$$\frac{1}{\tilde{W}(\omega)} = -\frac{1}{W(\omega)} - \frac{|\omega|}{2\pi} \quad (30)$$

and call what remain the dual impurity determinant. It is given by the same formula (17) with dual charge jump and  $\alpha \rightarrow \tilde{\alpha}$ .

After the cancelation of ballistic current expression for conductance becomes:

$$\mathfrak{G}(\omega) = \frac{1}{2\pi v_c(\omega)} |\mathcal{K}_\omega|^2, \quad (31)$$

where effective transition coefficient is given by the formula dual to (21)

$$\begin{aligned} &(2\pi)\delta(\omega - \omega') |\mathcal{K}_\omega|^2 = \\ &= \frac{i\pi v_c(\omega)}{|\omega| \tilde{W}(\omega) Z} \int \mathcal{D}\tilde{\alpha} \tilde{\mathfrak{D}}([\tilde{\alpha}], \omega) \tilde{\alpha}(-\omega') \times \\ &\times \tilde{\mathfrak{D}et}[\tilde{\alpha}]_{\text{imp}} \exp \left[ -\frac{1}{2} \int \frac{d\omega}{2\pi} \frac{\tilde{\alpha}(-\omega)\tilde{\alpha}(\omega)}{\tilde{W}(\omega)} \right]. \end{aligned} \quad (32)$$

To prove finally that two theories are dual we have to prove that charge jumps are equal:

$$\tilde{\mathfrak{D}}[\tilde{\alpha}](\omega) = \mathfrak{D}[\alpha](\omega) |_{\mathcal{R}, \alpha \leftrightarrow \mathcal{K}, \tilde{\alpha}}. \quad (33)$$

For this purpose we construct perturbations theory expansion of  $\mathfrak{D}(\omega)$  in  $\mathcal{R}$  (using (6)) and confront it to expansion of  $\tilde{\mathfrak{D}}(\omega)$  in  $\mathcal{K}$  (using (9)). Direct calculation (with resummation of the series) leads to the following result for attractive potential:

$$\begin{aligned} \int \frac{d\omega}{2\pi} \mathfrak{D}(\omega) e^{-i\omega t} &= -|\mathcal{R}|^2 \mathfrak{B}_1(t) + |\mathcal{R}|^2 \left\{ \left( \frac{|\mathcal{R}|}{|\mathcal{K}|} \right)^2 [\mathfrak{B}_2(t) + \right. \\ &\left. + \mathfrak{B}_3(t)] - \left( \frac{|\mathcal{R}|}{|\mathcal{K}|} \right)^4 [\mathfrak{B}_4(t) + \mathfrak{B}_5(t)] + \right. \end{aligned} \quad (34)$$

$$+ \left( \frac{|\mathcal{R}|}{|\mathcal{K}|} \right)^6 [\mathfrak{B}_6(t) + \mathfrak{B}_7(t)] \dots \Big\}.$$

Here coefficients  $\mathfrak{B}_n(t)$  are integrals

$$\begin{aligned} \mathfrak{B}_1(t) &= \frac{2}{\pi} \int \frac{d\tau}{(2\pi i)} \operatorname{Re} \left\{ \frac{\sin[\alpha(\tau) - \alpha(t)]}{\tau_{01}^2} \right\}, \\ \mathfrak{B}_2(t) &= \frac{2}{\pi} \int \frac{d^2\tau}{(2\pi i)^2} \frac{\sin[\alpha(\tau_1) - \alpha(\tau_2)]}{\tau_{01}\tau_{12}\tau_{02}}, \\ \mathfrak{B}_3(t) &= \\ &= \frac{2}{\pi} \int \frac{d^3\tau}{(2\pi i)^3} \frac{\sin[\alpha(\tau_1) - \alpha(\tau_2) + \alpha(\tau_3) - \alpha(t)]}{\tau_{01}\tau_{12}\tau_{23}\tau_{03}}, \quad (35) \\ \mathfrak{B}_4(t) &= \\ &= \frac{2}{\pi} \int \frac{d^4\tau}{(2\pi i)^4} \frac{\sin[\alpha(\tau_1) - \alpha(\tau_2) + \alpha(\tau_3) - \alpha(\tau_4)]}{\tau_{01}\tau_{12}\tau_{23}\tau_{34}\tau_{04}}, \end{aligned}$$

etc. Here  $\tau_{ik} = \tau_i - \tau_k - i\delta$  and  $\tau_0 \equiv t$ . For repulsive potential we obtain:

$$\begin{aligned} \int \frac{d\omega}{2\pi} \operatorname{sign} \omega \tilde{\mathfrak{D}}(\omega) e^{-i\omega t} &= |\mathcal{K}|^2 [\tilde{\mathfrak{B}}_1(t) + \tilde{\mathfrak{B}}_2(t)] - \\ &- |\mathcal{K}|^2 \left\{ \left( \frac{|\mathcal{K}|}{|\mathcal{R}|} \right)^2 [\tilde{\mathfrak{B}}_3(t) + \tilde{\mathfrak{B}}_4(t)] - \right. \\ &\left. - \left( \frac{|\mathcal{K}|}{|\mathcal{R}|} \right)^4 [\tilde{\mathfrak{B}}_5(t) + \tilde{\mathfrak{B}}_6(t)] + \dots \right\}, \quad (36) \end{aligned}$$

where coefficients  $\tilde{\mathfrak{B}}_n$  are obtained from  $\mathfrak{B}_n$  by substitution  $\alpha \rightarrow \tilde{\alpha}$ . It can be seen from (35) that even  $\mathfrak{B}_n$  can be expressed through odd ones (in momentum space):

$$\mathfrak{B}_{2n}(\omega) = -\theta(\omega) [2\mathfrak{B}_{2n-1}(\omega) + \mathfrak{B}_{2(n-1)}(\omega)], \quad n > 1.$$

For  $n = 1$  we have  $\mathfrak{B}_2(\omega) = -2\theta(\omega)\mathfrak{B}_1(\omega)$ . Hence:

$$\mathfrak{B}_{2n}(\omega) = 2\theta(\omega)(-1)^{n+1} \sum_{k=1}^n (-1)^k \mathfrak{B}_{2k-1}(\omega). \quad (37)$$

Identity (37) allows one to prove that coefficients of expansion in  $\mathcal{R}$  of Eq. (34) and expansion in  $\mathcal{K}$  of Eq. (36) coincide:

$$\begin{aligned} \mathfrak{B}_{2n-1}(\omega) + \mathfrak{B}_{2(n-1)}(\omega) &= \\ &= -\operatorname{sign}(\omega) [\mathfrak{B}_{2n}(\omega) + \mathfrak{B}_{2n-1}(\omega)]. \quad (38) \end{aligned}$$

This proves Eq. (33) and therefore the duality of the theory.

For point-like interaction dual effective potential  $\tilde{W}$  is equal to:

$$\tilde{W} = \frac{2\pi}{|\omega|} (v_c - 1) \quad (39)$$

so in this case duality can be formulated as substitution  $v_c \rightarrow v_c^{-1}$  and at small  $\mathcal{K}$  the effective transition coefficient is given by the formula dual to Eq. (26)

$$|\mathcal{K}_\omega|^2 = 2v_c \Gamma(-1 - 2\tilde{\nu}) \frac{\sin \pi \tilde{\nu}}{\pi} |\mathcal{K}|^2 \left( \frac{|\omega|}{M} \right)^{2\tilde{\nu}} \quad (40)$$

with  $\tilde{\nu} = v_c - 1$ . Again this formula is valid only at  $2\tilde{\nu} < 1$ , otherwise  $|\mathcal{K}_\omega|^2$  has a regular expansion in  $\omega$  starting from linear term.

Eqs. (26) and (40) are in line with results of [1] obtained in the first order of perturbation theory. We see that static conductance is zero for repulsive  $e-e$  interaction and tends to the ballistic one for attractive interaction. Analyzing the series (34) for effective reflection coefficient one can prove this statement for the point-like interaction in any order of perturbation theory. Details of this calculation will be published elsewhere.

The conductance  $\mathfrak{G}(\omega)$  for repulsing electrons should become ballistic one in the limit  $\mathcal{R} \rightarrow 0$  (analogously, for attracting electrons conductance should be zero at  $\mathcal{K} = 0$ ). For very small  $\mathcal{R}$  some transient region should exist, where  $\mathfrak{G}(\omega)$  is of order of unity. The width of this region can be estimated from the original formula (26) for the point-like interaction which gives expansion at small reflection coefficient:

$$|\mathcal{K}_\omega|^2 = 1 - \operatorname{const} |\mathcal{R}|^2 (M/\omega)^{2|\nu|} + \dots \quad (41)$$

Hence the transient region corresponds to

$$|\mathcal{R}| \sim (\omega/M)^{1-v_c^{-1}}.$$

Let us pay attention that power of  $\omega$  here is different from (40).

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