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Macroscopic approach to the Casimir friction force

V. V. Nesterenko[1)], *A. V. Nesterenko*

Bogoliubov Laboratory of Theoretical Physics, Joint Institute for Nuclear Research, 141980 Dubna, Russia

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The general formula is derived for the vacuum friction force between two parallel perfectly flat planes bounding two material media separated by a vacuum gap and moving relative to each other with a constant velocity $\mathbf{v}$. The material media are described in the framework of macroscopic electrodynamics whereas the nonzero temperature and dissipation are taken into account by making use of the Kubo formulae from non-equilibrium statistical thermodynamics. The formula obtained provides a rigorous basis for calculation of the vacuum friction force within the quantum field theory methods in the condensed matter physics. The revealed v-dependence of the vacuum friction force proves to be the following: for zero temperature ($T = 0$) it is proportional to $(v/c)^3$ and for $T > 0$ this force is linear in v/c.

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The existence of the vacuum friction force is widely discussed over past time. This force is considered to arise between two perfectly flat planes bounding material media separated by a vacuum gap and moving parallel to each other with a constant velocity. A variety of approaches applied to the study of this subject leads to contradictory results [1–5]. The issue on hand deals with a stationary, but irreversible process caused by the dissipation in polarizable media due to the vacuum friction. Hence, it is worthwhile to study this problem within the most general approach, i.e., by employing the macroscopic electrodynamics when describing material medium [6] and the Kubo formula for the linear response of the system to external action [7]. To the best of our knowledge such setting of the problem in question has not been proposed yet.

We consider a gap of width l between two solid half-spaces, call them 1 ($z < 0$) and 2 ($z > l$). We assume also that the half-space 1 is at rest in the laboratory reference frame, and the half-space 2 is moving with a constant velocity $\mathbf{v}$ which is parallel to the x axis $\mathbf{v} = (v, 0, 0)$. In what follows the consideration is conducted only in the laboratory rest frame.

We are interested in the electromagnetic field connected with the configuration described above. The corresponding Hamiltonian density is

$$w = \frac{1}{8\pi}\left(\mathbf{ED} + \mathbf{HB}\right). \tag{1}$$

The Gaussian units and the notations generally accepted in macroscopic electrodynamics are used [6]. The material relations [6]

$$\begin{aligned}\mathbf{D} + \frac{\mathbf{v}}{c}\times\mathbf{H} &= \varepsilon\left(\mathbf{E} + \frac{\mathbf{v}}{c}\times\mathbf{B}\right),\\ \mathbf{B} - \frac{\mathbf{v}}{c}\times\mathbf{E} &= \mu\left(\mathbf{H} - \frac{\mathbf{v}}{c}\times\mathbf{D}\right)\end{aligned} \tag{2}$$

enable one to express the displacements $\mathbf{D}$ and $\mathbf{B}$ in terms of the strength fields $\mathbf{E}$ and $\mathbf{H}$:

$$\begin{aligned}\mathbf{D} =& \varepsilon\mathbf{E} + \kappa\left[1 + \varepsilon\mu\left(\frac{\mathbf{v}}{c}\right)^2\right]\left(\frac{\mathbf{v}}{c}\times\mathbf{H}\right) -\\ &- \varepsilon\kappa\left[\frac{\mathbf{v}}{c}\left(\frac{\mathbf{v}}{c}\cdot\mathbf{E}\right) - \left(\frac{\mathbf{v}}{c}\right)^2\mathbf{E}\right],\\ \mathbf{B} =& \mu\mathbf{H} - \kappa\left[1 + \varepsilon\mu\left(\frac{\mathbf{v}}{c}\right)^2\right]\left(\frac{\mathbf{v}}{c}\times\mathbf{E}\right) -\\ &- \mu\kappa\left[\frac{\mathbf{v}}{c}\left(\frac{\mathbf{v}}{c}\cdot\mathbf{H}\right) - \left(\frac{\mathbf{v}}{c}\right)^2\mathbf{H}\right],\end{aligned} \tag{3}$$

where $\kappa = \varepsilon\mu - 1$. We restrict ourselves to the accuracy up to $(v/c)^3$ inclusively. As a result the Hamiltonian density (1) assumes the form:

$$\begin{aligned}w = w_0 &- \frac{\kappa}{c^2}\left[1 + \varepsilon\mu\left(\frac{\mathbf{v}}{c}\right)^2\right](\mathbf{v}\cdot\mathbf{S}) +\\ &+ \frac{\kappa}{8\pi}\left(\frac{v}{c}\right)^2\left[\varepsilon(E_y^2 + E_z^2) + \mu(H_y^2 + H_z^2)\right],\end{aligned} \tag{4}$$

where $\mathbf{S} = (c/4\pi)(\mathbf{E}\times\mathbf{H})$ is the Poynting vector and

$$w_0 = \frac{1}{8\pi}\left(\varepsilon\mathbf{E}^2 + \mu\mathbf{H}^2\right). \tag{5}$$

Obviously the terms in (4) depending on the velocity $\mathbf{v}$ of a medium concern only the half-space 2; in the case

[1)]e-mail: nestr@theor.jinr.ru

of the half-space 1 and the vacuum gap $0 < z < l$ these terms are absent. For simplicity we consider both media to be identical.

The motion of the half-spaces relative to each other will be treated as a weak perturbation that allows us to use the linear response theory [7]. In this approach, it is supposed that in the remote past ($t \to -\infty$) the perturbation was absent ($\mathbf{v} = 0$) and the whole system (electromagnetic field in both media 1, 2 and in the gap) was in thermodynamical equilibrium at the temperature T. Then the weak perturbation

$$\overline{w} = w - w_0 = -\frac{v}{c}\left[1 + \varepsilon\mu\left(\frac{v}{c}\right)^2\right]\frac{\kappa}{c}S_x + \\ + \frac{\kappa}{8\pi}\left(\frac{v}{c}\right)^2\left[\varepsilon(E_y^2 + E_z^2) + \mu(H_y^2 + H_z^2)\right] = \\ = \frac{v}{c}\left[1 + \varepsilon\mu\left(\frac{v}{c}\right)^2\right]w_1 + \left(\frac{v}{c}\right)^2 w_2 \tag{6}$$

is switched on adiabatically, so that the system does not go far away from the initial equilibrium state. When considering this state we are dealing with the electromagnetic field described by the Hamiltonian density w_0 at the temperature T. This field is connected with unbounded medium at rest possessing the vacuum gap of the width l made up by two parallel planes. Thus the unperturbed state is a standard Lifshitz configuration at the temperature T.

As the response of the system under study to the perturbation $\overline{w}$ we consider the ponderomotive force acting on the medium 2 along the x-axis. Obviously it is the vacuum friction force in the problem on hand. The density of this force

$$\sum_{\beta = x,y,z}\frac{\partial \sigma_{x\,\beta}(\mathbf{r})}{\partial r_\beta} \tag{7}$$

is given by the Maxwell stress tensor [6]

$$4\pi\sigma_{\alpha\beta} = E_\alpha D_\beta + H_\alpha B_\beta - \\ - \frac{\delta_{\alpha\beta}}{2}\left[(\mathbf{E}\cdot\mathbf{D}) + (\mathbf{H}\cdot\mathbf{B})\right], \quad \alpha, \beta = x, y, z. \tag{8}$$

For the sake of simplicity we use the Minkowski nonsymmetric form of the stress tensor $\sigma_{\alpha\beta}$ (see Refs. [6, 8]). This point is not crucial for us here. We shall turn to the Abraham symmetric energy-momentum tensor later.

Taking into account the geometry of the configuration under consideration it is enough to consider only the component

$$\sigma_{xz} = \frac{1}{4\pi}(E_x D_z + H_x B_z) \tag{9}$$

on the plane bounding the half-space 2, i.e., $\sigma_{xz}(\mathbf{r}_0)$, where $\mathbf{r}_0 = (x, y, z = l - 0)$. This quantity is the tangential strength in the x-direction exerted to the unit area of the surface $z = l - 0$.

By making use of the solution to the material relations (3) we obtain in the $(\mathbf{v}/c)^3$-approximation the following formula for σ_{xz}:

$$4\pi\,\sigma_{xz} = \left[1 + \kappa\left(\frac{v}{c}\right)^2\right](\varepsilon E_x E_z + \mu H_x H_z) + \\ + \kappa\,\frac{v}{c}\left[1 + \varepsilon\mu\left(\frac{v}{c}\right)^2\right](E_x H_y - E_y H_x) = \\ = \left[1 + \kappa\left(\frac{v}{c}\right)^2\right]\cdot 4\pi\,\sigma_{xz}^{(0)} + \\ + \frac{v}{c}\left[1 + \varepsilon\mu\left(\frac{v}{c}\right)^2\right]\cdot 4\pi\,\sigma_{xz}^{(1)}. \tag{10}$$

As in the Hamiltonian density (4), the terms depending on $\mathbf{v}$ in (10) are relative only to the medium 2.

Now we are in position to write out the general formula for the vacuum friction force in the framework of the Kubo formalism [7]

$$\langle\sigma_{xz}(\mathbf{r}_0)\rangle = \langle\sigma_{xz}(\mathbf{r}_0)\rangle_0 + \\ + \int\limits_{-\infty}^{+\infty} dt' \int d\mathbf{r}\,\langle\!\langle \sigma_{xz}(t, \mathbf{r}_0), \overline{w}(t', \mathbf{r})\rangle\!\rangle\,. \tag{11}$$

The integration over $d\mathbf{r}$ is carried out only in the half-space 2, i.e., for $z \geq l$ (cf. with Ref. [9]). Here $\langle\,...\,\rangle_0 = \mathrm{Tr}(\varrho_0\,...)$ denotes the averaging with the Gibbs statistical operator $\varrho_0 = \exp[(F - H_0)/kT]$, where

$$H_0 = \int d\mathbf{r}\,w_0(\mathbf{r}) \tag{12}$$

and the integration is carried out over the both media 1, 2 and over the vacuum gap; F is the corresponding free energy. The brackets $\langle\,...\,\rangle = \mathrm{Tr}(\varrho\,...)$ stand for the analogous averaging with the statistical operator ϱ which takes into account, in addition to w_0, the quantum-mechanical perturbation $\overline{w}$, the latter being treated in the linear approximation. The density operator ϱ obeys the condition $\varrho \to \varrho_0$, when $t \to -\infty$. The main object in (11) is the retarded Green function[2)] for two operators $A(t)$ and $B(t')$:

$$\langle\!\langle A(t), B(t')\rangle\!\rangle = \frac{1}{i\hbar}\theta(t - t')\big\langle[A(t), B(t')]\big\rangle_0. \tag{13}$$

The explicit time dependence of the operators $A(t)$ and $B(t')$ in (13) implies the Heisenberg representation with the Hamiltonian H_0 from (12).

[2)]Kubo [7] used the response function $\varphi_{AB}(t - t')$ which is related to the Green function in a simple way $\langle\!\langle A(t), B(t')\rangle\!\rangle = -\theta(t - t')\varphi_{AB}(t - t')$.

It is easy to show that the first term in the right-hand side of (11) vanishes $\langle\sigma(\mathbf{r}_0)\rangle_0 = 0$. Indeed

$$4\pi\,\sigma_{xz}^{(0)} = \varepsilon E_x E_z + \mu H_x H_z = 4\pi\sigma_{xz}\big|_{\mathbf{v}=0}, \qquad (14)$$

$$4\pi\,\sigma_{xz}^{(1)} = \kappa(E_x H_y - E_y H_x)\,. \qquad (15)$$

At the equilibrium state the expected values of all components of the Maxwell stress tensor should vanish, hence $\langle\sigma_{xz}^{(0)}\rangle_0 = 0$. The value of $\langle\sigma_{xz}^{(1)}\rangle_0$ also vanishes because $\mathbf{E}(t,\mathbf{r})$ and $\mathbf{H}(t,\mathbf{r})$ are not correlated at the same time t and point $\mathbf{r}$ [10].

Now we can simplify the integral term in (11). We take into account the fact that only the terms of the odd power in v/c can be put down to the friction force. As a result we obtain in the $(v/c)^3$-approximation

$$\langle\sigma_{xz}(\mathbf{r}_0)\rangle = \frac{v}{c}\left[1 + (\kappa+\varepsilon\mu)\left(\frac{v}{c}\right)^2\right]\times$$
$$\times\int\limits_{-\infty}^{+\infty} dt'\int d\mathbf{r}\,\left\langle\left\langle\sigma_{xz}^{(0)}(t,\mathbf{r}_0), w_1(t',\mathbf{r})\right\rangle\right\rangle +$$
$$+\left(\frac{v}{c}\right)^3\int\limits_{-\infty}^{+\infty} dt'\int d\mathbf{r}\,\left\langle\left\langle\sigma_{xz}^{(1)}(t,\mathbf{r}_0), w_2(t',\mathbf{r})\right\rangle\right\rangle, \qquad (16)$$

where $\sigma_{xz}^{(0)}$, $\sigma_{xz}^{(1)}$ are defined in (10), (14), (15) and w_1, w_2 are introduced in (6) and read

$$w_1 = -\frac{\kappa}{c}S_x, \quad w_2 = \frac{\kappa}{8\pi}\Big[\varepsilon(E_y^2+E_z^2) + \mu(H_y^2+H_z^2)\Big]. \qquad (17)$$

Let us prove that the first term in (16) vanishes at zero temperature. First of all it is to be noted that at $T=0$ and $\mathbf{v}=0$ the electromagnetic field in the half-space 2 can be considered as isolated system with its conserved total momentum P_x:

$$P_x = \frac{1}{c}\int\limits_{z\geq l} d\mathbf{r}\,T^{x0}(\mathbf{r}) = \frac{1}{c}\int\limits_{z\geq l} d\mathbf{r}\,T^{0x}(\mathbf{r}) = \frac{1}{c^2}\int\limits_{z\geq l} d\mathbf{r}\,S_x(\mathbf{r}). \qquad (18)$$

Here we have assumed that $T^{x0} = T^{0x}$ are the components of the Abraham symmetric energy-momentum tensor [8, 11]. With regards for this the commutator entering the first term in (16) gives (see, for example [12]):

$$\int\limits_{z\geq l} d\mathbf{r}\big\langle[\sigma_{xz}^{(0)}(t,\mathbf{r}_0), S_x(t',\mathbf{r})]\big\rangle_0 = c^2\big\langle[\sigma_{xz}^{(0)}(t,\mathbf{r}_0), P_x]\big\rangle_0 =$$
$$= c^2\frac{\hbar}{i}\frac{\partial}{\partial x}\big\langle\sigma_{xz}^{(0)}(t,\mathbf{r}_0)\big\rangle_0 = 0. \qquad (19)$$

In this formula the averaging $\langle\,...\,\rangle_0$ is carried out with respect to the lowest energy state, i.e., with respect to the quantum-field vacuum state.

For $T>0$ and $\mathbf{v}=0$ this reasoning does not hold because in this case the electromagnetic field in the half-space 2 interacts with the black body radiation filling the gap $0<z<l$. It is this interaction that ensures the equilibrium of electromagnetic field in half-spaces 1, 2 and in the gap.

Thus for $T>0$ the vacuum friction force is defined by the first term in (16) which is linear in v/c. For $T=0$ this force is described by the second term in (16) which is proportional to $(v/c)^3$. It should be noted that in our approach this dependence of the Casimir friction force on the relative velocity has, as a matter of fact, the kinematical reason. In Refs. [13–15] such v-dependence of the vacuum friction force was obtained in a simple quantum mechanical models.

Further calculations demand construction of the four-point Green functions entering final formula (16), i.e., expressing them in terms of the basic two-point retarded Green function of electromagnetic field in a medium [10]. This procedure can wittingly be brought about for linear dielectrics [16]. However its realization and removing the divergencies is a nontrivial task which requires development of special technique and approximation. All this is beyond the scope of the present paper.

Closing we would like to stress the following. The final formula (16) provides a rigorous basis for calculation of the vacuum friction force in the framework of the quantum field theory methods in the condensed matter physics. The revealed v-dependence of the vacuum friction force proves to be the following: for zero temperature ($T=0$) it is proportional to $(v/c)^3$ and for $T>0$ this force is linear in v/c.

It is also important to note that the Green functions in resulting formula (16) involve only unperturbed electric ($\mathbf{E}$) and magnetic ($\mathbf{H}$) fields governed by "free" Hamiltonian H_0 describing the standard Lifshitz configuration. Thus in our approach, unlike other considerations of this problem (see, for example, [4] and references therein) there is no need to solve the Maxwell equations with moving boundaries.

Another obvious advantage of the proposed approach to the calculation of the Casimir friction force is a correct treatment, from the very beginning, of the relativistic invariance in this problem [14]. It is due to the employment of the Minkowski material relation (2).

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