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Kazama–Suzuki models of $N = 2$ superconformal field theory and Manin triples

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Kazama–Suzuki coset models is an interesting class of $N = 2$ supersymmetric models of conformal field theory which are used to build realistic models of superstring in 4 dimensions. We formulate Kazama–Suzuki construction of $N = 2$ superconformal coset models using more general language of Manin triples and represent the corresponding $N = 2$ Virasoro superalgebra currents in explicit form.

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It is by now well-known due to Gepner [1] that the unitary $N = 2$ superconformal field theories play an important role in the construction of realistic models of superstring compactification from 10 to 4 dimensions. The idea of his construction is to use $N = 2$ superconformal field theories with central charge 9 for the internal sector of the string degrees of freedom. In particular, Gepner considered a product of $N = 2$ minimal models such that their total central charges adds up to 9. Then he showed that GSO projection can be obtained by selecting the states with odd integral $U(1)$ charges and this projection can be made consistent with modular invariance if one adds at the same time the spectral flow twisted sectors (see [1]).

In the paper [2] Kazama and Suzuki constructed a new large class of $N = 2$ superconformal field theories, including $N = 2$ superconformal minimal models, using the coset space method [3]. It rised the problem of generalization of the Gepner's construction to include these new models and investigate the corresponding models of the superstring compactification from 10 to 4 dimensions. However this problem is quite far from its solution untill now mainly because of the Kazama–Suzuki models themselves has not yet been studied quite well.

As a particular case of Kazama–Suzuki models $N = 2$ superconformal WZNW models was investigated in [4] and [5] (see also [6]) where a one to one correspondence between the Manin triples, which are the classical limit of Drinfeld's quantum dubles [7] and $N = 2$ superconformal WZNW models was established. Since then it has become clear that this important relation should also takes place for $N = 2$ superconformal coset models and Kazama–Suzuki construction can be formulated naturally in terms of Manin triples also. However by some of the reasons the explicit formulas has not been represented in the literature in spite of its importance at least in the context of Poisson–Lie T-duality [8] and Mirror Symmetry [9, 10] of the superstring vaccua.

In this note we fill up this defect formulating Kazama–Suzuki construction on the language of Manin triples and represent the corresponding $N = 2$ Virasoro superalgebra currents entirely in terms of Manin triple and Manin subtriple objects. We begin with the definition of Manin triple [7].

Definition. A Manin triple (g, g_+, g_-) consists of a Lie algebra g, with nondegenerate invariant inner product $(.,.)$ and isotropic Lie subalgebras $g_\pm$ such that $g = g_+ \oplus g_-$ as a vector space.

For any finite dimensional Manin triple let us fix arbitrary orthonormal basis $\{E^A, E_A,\ A = 1, ..., D\}$ in algebra g so that $\{E^A\}$ is a basis in g_+, $\{E_A\}$ is a basis in g_-. The commutators and Jacobi identity of g have the form

$$\begin{gathered}[E^A, E^B] = f^{AB}_C E^C,\\ [E_A, E_B] = f^C_{AB} E_C,\\ [E^A, E_B] = f^A_{BC} E^C - f^{AC}_B E_C,\end{gathered} \tag{1}$$

$$\begin{gathered}f^{AB}_D f^{DC}_E + f^{BC}_D f^{DA}_E + f^{CA}_D f^{DB}_E = 0,\\ f^D_{AB} f^E_{DC} + f^D_{BC} f^E_{DA} + f^D_{CA} f^E_{DB} = 0,\\ f^A_{MC} f^{BM}_D - f^A_{MD} f^{BM}_C - f^B_{MC} f^{AM}_D + f^B_{MD} f^{AM}_C =\\ = f^M_{CD} f^{AB}_M.\end{gathered} \tag{2}$$

In what follows the following consequence of (2)

$$f_M f^{MB}_A + f^M f^B_{MA} = f^B_{NM} f^{NM}_A, \tag{3}$$

where $f_M = f^A_{MA}$, $f^M = f^{MA}_A$, will be important to us.

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Denote by $\langle .,. \rangle$ the Killing form of g. It is not difficult to find

$$\begin{gathered}\langle E^A, E^B\rangle = 2f^{AC}_D f^{BD}_C,\\ \langle E_A, E_B\rangle = 2f^D_{AC} f^C_{BD},\\ \langle E^A, E_B\rangle = -f^{CD}_B f^A_{CD} - 2f^{AC}_D f^D_{BC},\end{gathered} \tag{4}$$

and rewrite the last expression from (3) in the form

$$\langle E^B, E_C\rangle = -K^B_C - 2A^B_C, \tag{5}$$

where

$$\begin{gathered}K^B_A = f_C f^{CB}_A + f^C f^B_{CA},\\ A^B_C = f^D_{CM} f^{BM}_D.\end{gathered} \tag{6}$$

Let $J^A(z), J_A(z)$ be the generators of the affine Kac–Moody algebra $\hat{g}$, that correspond to the fixed basis $\{E^A, E_A\}$, so that currents J^A generate subalgebra $\hat{g}_+$ and currents J_A generate subalgebra $\hat{g}_-$. The singular OPEs between these currents is the following

$$\begin{gathered}J^A(z)J^B(w) = -(z-w)^{-2} \times\\ \times \frac{1}{2}\langle E^A, E^B\rangle + (z-w)^{-1} f^{AB}_C J^C(w) + \text{reg},\\ J_A(z)J_B(w) = -(z-w)^{-2} \times\\ \times \frac{1}{2}\langle E_A, E_B\rangle + (z-w)^{-1} f^C_{AB} J_C(w) + \text{reg},\\ J^A(z)J_B(w) = (z-w)^{-2} \cdot \frac{1}{2}(q\delta^A_B - \langle E^A, E_B\rangle) +\\ + (z-w)^{-1}(f^A_{BC} J^C - f^{AC}_B J_C)(w) + \text{reg},\end{gathered} \tag{7}$$

where q is a number.

Let $\psi^A(z), \psi_A(z)$ be free fermion currents which have singular OPEs with respect to the inner product (.,.)

$$\psi^A(z)\psi_B(w) = (z-w)^{-1}\delta^A_B + \text{reg}. \tag{8}$$

It has been shown in [5] that the currents

$$\begin{gathered}G^+ = \frac{2}{\sqrt{q}}\left(\psi^A J_A - \frac{1}{2} f^C_{AB} :\psi^A\psi^B\psi_C:\right),\\ G^- = \frac{2}{\sqrt{q}}\left(\psi_A J^A - \frac{1}{2} f^{AB}_C :\psi_A\psi_B\psi^C:\right),\\ K = \left(\delta^B_A + \frac{2}{q}K^B_A\right) :\psi^A\psi_B: + \frac{2}{q}(f_C J^C - f^C J_C),\\ T = \frac{1}{q} :(J^A J_A + J_A J^A): + \frac{1}{2} :(\partial\psi^A\psi_A - \psi^A\partial\psi_A):\end{gathered} \tag{9}$$

satisfy the $N=2$ Virasoro superalgebra OPE's:

$$\begin{gathered}T(z_1)T(z_2) = z_{12}^{-4}\frac{c}{2} + z_{12}^{-2}\cdot 2T(z_2) + z_{12}^{-1}\partial T(z_2) + \text{reg.},\\ T(z_1)K(z_2) = z_{12}^{-2}K(z_2) + z_{12}^{-1}\partial K(z_2) + \text{reg.},\\ T(z_1)G^\pm(z_2) = z_{12}^{-2}\cdot\frac{3}{2}G^\pm(z_2) + z_{12}^{-1}\partial G^\pm(z_2) + \text{reg.},\\ K(z_1)K(z_2) = z_{12}^{-2}\frac{c}{3} + \text{reg.},\\ K(z_1)G^\pm(z_2) = \pm z_{12}^{-1}G^\pm(z_2) + \text{reg.},\\ G^+(z_1)G^-(z_2) = z_{12}^{-3}\frac{2c}{3} + z_{12}^{-2}\cdot 2K(z_2) +\\ + z_{12}^{-1}[2T(z_2) + \partial K(z_2)] + \text{reg.},\\ G^+(z_1)G^+(z_2) = \text{reg.},\quad G^-(z_1)G^-(z_2) = \text{reg.},\end{gathered} \tag{10}$$

where the central charge

$$c = 3\left(D + \frac{2}{q}K^C_C\right) \tag{11}$$

and $z_{12} = z_1 - z_2$.

Let us fix some Manin subtriple

$$(h, h_+, h_-) \subset (g, g_+, g_-) \tag{12}$$

such that h_+ is spanned by the subset $\{E^\alpha\}$, $\alpha = 1, ..., D-d$ of the generators $\{E^A\}$, and h_- is spanned by the subset $\{E^\alpha\}$, $\alpha = 1, ..., D-d$ of the generators $\{E_A\}$. In what follows we use the small Latin letters to denote the indexes for the remaining generators E^a and E_a which span $t_+ = g_+/h_+$ and $t_- = g_-/h_-$ correspondingly.

The Manin triples formulation of the conditions found by Kazama and Suzuki [2] is given by the following

Theorem. Suppose the subspaces t_+ and t_- are subalgebras. Then the currents

$$G^+ = \frac{2}{\sqrt{q}}\left(\psi^a J_a - \frac{1}{2} f^c_{ab} : \psi^a\psi^b\psi_c :\right),$$
$$G^- = \frac{2}{\sqrt{q}}\left(\psi_a J^a - \frac{1}{2} f^{ab}_c : \psi_a\psi_b\psi^c :\right),$$
$$K = \left(\delta^b_a + \frac{2}{q}Q^b_a\right) : \psi^a\psi_b : + \frac{2}{q}\left(\phi_C J^C - \phi^C J_C\right),$$
$$\begin{aligned}T = \frac{1}{q} : (J_a J^a + J^a J_a) : &+ \\ + \frac{2}{q}(f^b_{\mu a}J^\mu - f^{\mu b}_a J_\mu) : \psi^a\psi_b : &+ \\ + \frac{1}{2q}(f^c_{\mu a}f^{d\mu}_b - f^c_{\mu b}f^{d\mu}_a - f^d_{\mu a}f^{c\mu}_b &+ \\ + f^d_{\mu b}f^{c\mu}_a) : \psi^a\psi^b\psi_c\psi_d : &+ \\ + \frac{1}{2}\left[\delta^b_a + \frac{2}{q}(f^b_{\mu n}f^{\mu n}_a + f^n_{\mu a}f^{\mu b}_n)\right] : (\partial\psi^a\psi_b - \psi^a\partial\psi_b) :\end{aligned} \tag{13}$$

where

$$\begin{aligned}Q^b_a &= \phi_M f^{Mb}_a + \phi^M f^b_{Ma},\\ \phi_M &= f^b_{Mb}, \quad \phi^M = f^{Mb}_b\end{aligned} \tag{14}$$

generate the $N=2$ Virasoro superalgebra (10) with the central charge

$$c = 3\left(d + \frac{2}{q}Q^a_a\right). \tag{15}$$

The statement of the theorem is more or less obvious if one notices that $G^\pm(z)$ are the BRST currents of $t_\pm$ subalgebras so the nilpotence of the currents follows from the OPE's for the currents $J^a(z)$ and $J_a(z)$ correspondingly. To check the remaining $N=2$ superalgebra Virasoro OPE's one can just follow the analysis of Kazama and Suzuki.

It make sense however to prove the theorem by the direct calculation of the operator product expansions of the currents (13) and using some important identities the structure constants of Manin triple (g, g_+, g_-) and Manin subtriple (h, h_+, h_-) satisfy according to the assumption. First of all we note that

$$f^{\mu a}_\nu = f^\nu_{\mu a} = 0 \tag{16}$$

because of $t_\pm$ are isotropic subalgebras and the form $(\ldots)$ is invariant and nondegenerate. Due to (7), (8) and (16) we obtain

$$\begin{aligned}G^+(z_1)G^-(z_2) = {}& z^{-3}_{12}\left(2d - \frac{2}{q}\langle E^a, E_a\rangle - \frac{2}{q}f^c_{ab}f^{ab}_c\right) + \\ &+ z^{-2}_{12}\Big\{\left[2\delta^b_a - \frac{2}{q}\langle E^b, E_a\rangle + \frac{2}{q}(2f^d_{ac}f^{cb}_d + f^b_{cd}f^{cd}_a)\right]\times \\ &\times : \psi^a\psi_b : (z_2) + \frac{4}{q}(\phi_M J^M - \phi^M J_M)(z_2)\Big\} + \\ &+ z^{-1}_{12}\Big[\frac{2}{q}(: J_a J^a : + : J^a J_a :)(z_2) + \\ &+ \frac{2}{q}(\phi_M\partial J^M - \phi^M\partial J_M)(z_2) + \\ &+ \frac{4}{q}(f^b_{\mu a}J^\mu - f^{\mu b}_a J_\mu) : \psi^a\psi_b : (z_2) + \\ &+ \frac{1}{q}(f^n_{ab}f^{cd}_n - f^c_{an}f^{nd}_b + f^c_{bn}f^{nd}_a + f^c_{an}f^{dn}_b - f^d_{an}f^{cn}_b)\times \\ &\times : \psi^a\psi^b\psi_c\psi_d : (z_2) + \\ &+ \left(2\delta^b_a - \frac{2}{q}\langle E^b, E_a\rangle + \frac{4}{q}f^d_{ac}f^{cb}_d\right)\partial\psi^a\psi_b + \\ &+ \frac{2}{q}f^b_{cd}f^{cd}_a\psi^a\partial\psi_b(z_2)\Big] + \text{reg.}\end{aligned} \tag{17}$$

Using (4) and (16) we find that

$$\begin{aligned}&-\langle E^b, E_a\rangle + 2f^d_{ac}f^{cb}_d + f^b_{cd}f^{cd}_a = \\ &= 2(f^{\mu n}_a f^b_{\mu n} + f^{b\mu}_n f^n_{a\mu} + f^b_{nm}f^{nm}_a).\end{aligned} \tag{18}$$

Then one can show using the last identity from (2) that

$$f^{\mu n}_a f^b_{\mu n} + f^{b\mu}_n f^n_{a\mu} + f^b_{nm}f^{nm}_a = \phi_M f^{Mb}_a + \phi^M f^b_{Ma} = Q^b_a. \tag{19}$$

Similarly we find

$$\begin{aligned}&f^n_{ab}f^{cd}_n - f^c_{an}f^{nd}_b + f^c_{bn}f^{nd}_a + f^c_{an}f^{dn}_b - f^d_{an}f^{cn}_b = \\ &= f^c_{\mu a}f^{d\mu}_b - f^c_{\mu b}f^{d\mu}_a - f^d_{\mu a}f^{c\mu}_b + f^d_{\mu b}f^{c\mu}_a.\end{aligned} \tag{20}$$

It gives

$$\begin{aligned}G^+(z_1)G^-(z_2) = {}& z^{-3}_{12}\cdot 2\left(d + \frac{2}{q}Q^a_a\right) + \\ &+ z^{-2}_{12}\cdot 2\left(\delta^b_a + \frac{2}{q}Q^b_a\right) :\psi^a\psi_b: + \frac{2}{q}(\phi_C J^C - \phi^C J_C)(z_2) + \\ &+ z^{-1}_{12}\Big\{\frac{2}{q}:(J_a J^a + J^a J_a): + \frac{4}{q}(f^b_{\mu a}J^\mu - f^{\mu b}_a J_\mu):\psi^a\psi_b: + \\ &+ \frac{1}{q}(f^c_{\mu a}f^{d\mu}_b - f^c_{\mu b}f^{d\mu}_a - f^d_{\mu a}f^{c\mu}_b + f^d_{\mu b}f^{c\mu}_a):\psi^a\psi^b\psi_c\psi_d: + \\ &+ \left[\delta^b_a + \frac{2}{q}(f^b_{\mu n}f^{\mu n}_a + f^n_{\mu a}f^{\mu b}_n)\right] : (\partial\psi^a\psi_b - \psi^a\partial\psi_b) : + \\ &+ \left(\delta^b_a + \frac{2}{q}Q^b_a\right)\partial(: \psi^a\psi_b :) + \\ &+ \frac{2}{q}(\phi_C\partial J^C - \phi^C\partial J_C)\Big\} + \text{reg.}\end{aligned} \tag{21}$$

Thus we identify the central charge c, $U(1)$ current $K(z)$ and stress-energy tensor $T(z)$ according to (13), (15).

Now we calculate the OPE

$$\frac{\sqrt{q}}{2}K(z_1)G^+(z_2)=z_{12}^{-2}\frac{2}{q}\times$$
$$\times\left(Q^n_kf^k_{nm}+\frac{1}{2}\langle\phi^ME_M-\phi_ME^M,E_n\rangle\right)\psi^m(z_2)+$$
$$+z_{12}^{-1}\frac{2}{q}\Big[Q^b_a\psi^aJ_b-(\phi_Mf^{MB}_a+\phi^Mf^B_{Ma})\times$$
$$\times\psi^aJ_B+\phi_Mf^M_{aB}\psi^aJ^B+$$
$$+\left(\frac{1}{2}Q^c_nf^n_{ab}-Q^n_af^c_{nb}\right)\psi^a\psi^b\psi_c\Big](z_2)+$$
$$+z_{12}^{-1}\frac{\sqrt{q}}{2}G^+(z_2)+\text{reg.}\qquad(22)$$

It can be shown that

$$\phi_Mf^M_{aN}=0,\quad\phi^Mf^{aN}_M=0,\qquad(23)$$

$$\phi_Mf^{\mu M}_a=0,\quad\phi^Mf^a_{\mu M}=0\qquad(24)$$

because of (2) and (16). Hence,

$$\frac{\sqrt{q}}{2}K(z_1)G^+(z_2)=z_{12}^{-2}\times$$
$$\times\frac{2}{q}\left(Q^n_kf^k_{nm}+\frac{1}{2}\langle\phi^ME_M-\phi_ME^M,E_n\rangle\right)\psi^m(z_2)+$$
$$+z_{12}^{-1}\frac{2}{q}\left(\frac{1}{2}Q^c_nf^n_{ab}-Q^n_af^c_{nb}\right)\psi^a\psi^b\psi_c(z_2)+$$
$$+z_{12}^{-1}\frac{\sqrt{q}}{2}G^+(z_2)+\text{reg.}\qquad(25)$$

Because of Jacobi identity for t_--subalgebra we obtain

$$Q^c_nf^n_{ab}-Q^n_af^c_{nb}+Q^n_bf^c_{na}=$$
$$=\phi_M(f^{Mc}_nf^n_{ab}-f^{Mn}_af^c_{nb}+f^{Mn}_bf^c_{na}).\qquad(26)$$

But the right hand side of this expression is vanishing due to (2) and the first identity from (24) so that the only unwanted term in the right hand side of (22) is the second pole contribution. This contribution is vanishing also because

$$\frac{1}{2}\langle\phi^ME_M-\phi_ME^M,E_m\rangle=$$
$$=\phi^Mf^n_{Mk}f^k_{mn}+\phi_Mf^{Mn}_kf^k_{mn}=-Q^n_kf^k_{nm},\qquad(27)$$

where the (4), the first identity from (24) and the definition of Q^a_b have been used. Thus we obtain

$$K(z_1)G^+(z_2)=z_{12}^{-1}G^+(z_2)+\text{reg.}\qquad(28)$$

It can be shown similarly that the remaining OPE's from (10) are satisfied also. It finishes the proof.

Notice that because of $t_\pm$ are subalgebras, the Manin subtriple (h,h_+,h_-) makes corresponding subrgoup H the Poisson-Lie subgroup of the Poisson–Lie group G, such that the homogeneous space G/H becomes Poisson homogeneous due to Drinfeld [12] and Semenov-Tian-Shansky [13]. However the Poisson structure on G/H is special [12] so it would be interesting to see if one can generalize the Manin triple construction of $N=2$ superconformal coset models to include more general Poisson homogeneous G/H spaces described in [12].

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