On the question of the Wilson fermion doubling phenomenon on irregular lattice

S. N. Vergeles[1)]

Landau Institute for Theoretical Physics of the RAS, 142432 Chernogolovka, Russia

Department of Theoretical Physics, Moscow Institute of Physics and Technology, 141700 Dolgoprudnyj, Russia

Submitted 15 September 2014
Resubmitted 30 September 2014

It is shown that the Wilson fermion doubling phenomenon on irregular lattices (simplicial complexes) does exist. This means that the irregular (not smooth) zero or soft modes exist. The statement is proved on 4-Dimensional lattice by means of the Atiyah–Singer index theorem.

DOI: 10.7868/S0370274X14210115

1. The Wilson fermion doubling phenomenon on the regular periodic lattices has been discovered long ago in [1]. The phenomenon and its influence on physics was studied in a number of works (for example see [2–4]). It was proved in [5, 6] that the fermion doubling phenomenon indeed takes place on any periodic lattice with local fermion action transforming to the usual Dirac action in long-wavelength region. However, the question about the existence of the Wilson fermion doubling on irregular lattices is open at present. This means that the problem is unsolved in the case of lattice quantum gravity theory [2)] (see [7, 8]).

In this letter I show that the Wilson fermion doubling phenomenon on irregular 4 Dimensional lattices (simplicial complexes) does exist. However, there exists a fundamental difference between the propagation of doubling modes on regular and irregular lattices. In the first case the propagator of the irregular modes is the same as the propagator of the regular modes from the spectrum origin, i.e. power-behaved. On the contrary, the propagation of irregular modes on irregular lattice is similar to the Markov process of a random walks. Thus the propagator of irregular modes on irregular lattice decreases very quickly: the doubled irregular modes are "bad"quasiparticles. The detailed investigation of the problem will be made in the subsequent work.

2. First of all, one must outline shortly the Dirac system on the simplicial complexes. More general problem (the definition of Dirac system in discrete lattice gravity) has been solved in [7, 8]. Here I simplify the problem assuming that the 4-Dimensional simplicial complex $\mathfrak{K}$ is embedded into 4-Dimensional Euclidean space and the curvature and torsion are equal to zero.

Further all definitions and designations are similar to that in [7, 8]. The four Dirac matrices (4×4) satisfy the well known Clifford algebra

$$\gamma^a\gamma^b + \gamma^b\gamma^a = 2\delta^{ab}, \quad \gamma^5 \equiv \gamma^1\gamma^2\gamma^3\gamma^4\,,$$
$$\mathrm{tr}\,\gamma^5\gamma^a\gamma^b\gamma^c\gamma^d = 4\,\varepsilon^{abcd}, \quad \sigma^{ab} \equiv \frac{1}{4}\left[\gamma^a,\,\gamma^b\right]. \qquad (1)$$

The vertices of the complex are denoted as $a_{\mathcal{V}}$, the index $\mathcal{V} = 1, 2, \ldots, \mathfrak{N} \to \infty$ enumerates the vertices. Let the index $\mathcal{W}$ enumerates 4-simplices. It is necessary to use the local enumeration of the vertices $a_{\mathcal{V}}$ attached to a given 4-simplex: the all five vertices of a 4-simplex with index $\mathcal{W}$ are enumerated as $a_{\mathcal{W}i}$, $a_{\mathcal{W}j}$, $a_{\mathcal{W}k}$, $a_{\mathcal{W}l}$, and $a_{\mathcal{W}m}$, $i,\,j,\ldots = 1, 2, 3, 4, 5$. It must be kept in mind that the same vertex, 1-simplex et cetera can belong to the another adjacent 4-simplexes. Later the notations with extra index $\mathcal{W}$ indicate that the corresponding quantities are assigned to elements of the complex $\mathfrak{K}$, moreover these elements belong to the 4-simplex with index $\mathcal{W}$. The Levi–Civita symbol with in pairs different indexes $\varepsilon_{\mathcal{W}ijklm} = \pm 1$ depending on whether the order of vertices $a_{\mathcal{W}i}a_{\mathcal{W}j}a_{\mathcal{W}k}a_{\mathcal{W}l}a_{\mathcal{W}m}$ defines the positive or negative orientation of this 4-simplex. For each oriented 1-simplex $a_{\mathcal{W}i}a_{\mathcal{W}j}$ of the 4-dimensional simplicial complex $\mathfrak{K}$ an element of the (isotopic) group $SU(2)$

$$U_{\mathcal{W}ij} = U^{-1}_{\mathcal{W}ji} = \exp\left(ieA_{\mathcal{W}ij}\right), \quad A_{\mathcal{W}ij} \in \mathcal{L}, \qquad (2)$$

where $\mathcal{L}$ is the Lie algebra of the group $SU(2)$, and an elementary vector

$$e^a_{\mathcal{W}ij} \equiv -e^a_{\mathcal{W}ji}, \qquad (3)$$

[1)] e-mail: vergeles@itp.ac.ru

[2)] All variants of the lattice gravity theory are defined on the simplicial complexes.

are assigned. The Dirac spinors $\psi_{\mathcal{V}}$ and $\psi^{\dagger}_{\mathcal{V}}$ are assigned to each vertex $a_{\mathcal{V}}$. The Dirac spinors and the gauge field $A_{\mathcal{W}ij}$ belong to the same representation of algebra $\mathcal{L}$.

The Euclidean Hermitean action of the Dirac field associated with the complex $\mathfrak{K}$ has the form

$$\mathfrak{A}_{\psi} = -\frac{1}{5\times 6\times 24}\sum_{\mathcal{W}}\sum_{i,j,k,l,m}\varepsilon_{\mathcal{W}ijklm}\varepsilon^{abcd}\times$$
$$\times\left(i\,\psi^{\dagger}_{\mathcal{W}m}\gamma^{a}U_{\mathcal{W}mi}\psi_{\mathcal{W}i}\right)e^{b}_{\mathcal{W}mj}e^{c}_{\mathcal{W}mk}e^{d}_{\mathcal{W}ml} =$$
$$= \sum_{\mathcal{V}_1\mathcal{V}_2}\psi^{\dagger}_{\mathcal{V}_1}\left[-i\mathcal{D}_{\mathcal{V}_1,\mathcal{V}_2}\right]\psi_{\mathcal{V}_2}. \tag{4}$$

The action (4) is invariant under the gauge transformations

$$U_{\mathcal{W}ij}\to S_{\mathcal{W}i}\,U_{A\,ij}\,S^{-1}_{\mathcal{W}j},\quad S_{\mathcal{W}i}\in SU(2),$$
$$\psi_{\mathcal{W}i}\to S_{\mathcal{W}i}\,\psi_{\mathcal{W}i},\qquad \psi^{\dagger}_{\mathcal{W}i}\to\psi^{\dagger}_{\mathcal{W}i}\,S^{-1}_{\mathcal{W}i}. \tag{5}$$

The curvature in (4) is equal to zero by definition. The system of equations

$$e^{a}_{ij}+e^{a}_{jk}+\ldots+e^{a}_{li}=0 \tag{6}$$

means that the torsion is also zero. Here the sums in the parentheses are taken on any and all closed paths. Therefore the following interpretation is valid: $e^{a}_{\mathcal{W}ij} = \left(x^{a}_{\mathcal{W}j}-x^{a}_{\mathcal{W}i}\right)$, where $x^{a}_{\mathcal{W}i}$ are the cartesian coordinates of the vertex $a_{\mathcal{W}i}$.

Let

$$v_{\mathcal{W}} = \frac{1}{(4!)(5!)}\varepsilon_{abcd}\varepsilon_{\mathcal{W}ijklm}e^{a}_{\mathcal{W}mi}e^{b}_{\mathcal{W}mj}e^{c}_{\mathcal{W}mk}e^{d}_{\mathcal{W}ml} \tag{7}$$

be the oriented volume of the $\mathcal{W}$-4-simplex and $v_{\mathcal{V}}$ be the sum of the volumes $v_{\mathcal{W}}$ for that $\mathcal{W}$-4-simplexes which contain the vertex $a_{\mathcal{V}}$. Thus the spinor space scalar product is given by

$$\langle\psi_1|\psi_2\rangle = \frac{1}{5}\sum_{\mathcal{V}}v_{\mathcal{V}}\psi^{\dagger}_{(1)\mathcal{V}}\psi_{(2)\mathcal{V}}. \tag{8}$$

The operator $[i\mathcal{D}_{\mathcal{V}_1,\mathcal{V}_2}]$ in (4), as well as the operator $\left[i\,(v_{\mathcal{V}_1})^{-1/2}\,\mathcal{D}_{\mathcal{V}_1,\mathcal{V}_2}\,(v_{\mathcal{V}_2})^{-1/2}\right]$, are Hermitian. Thus the eigenfunction problem

$$\sum_{\mathcal{V}_2}\left[i\left(\frac{1}{\sqrt{v_{\mathcal{V}_1}}}\right)\mathcal{D}_{\mathcal{V}_1,\mathcal{V}_2}\left(\frac{1}{\sqrt{v_{\mathcal{V}_2}}}\right)\right]\left(\sqrt{v_{\mathcal{V}_2}}\psi_{(\mathfrak{P})\mathcal{V}_2}\right) =$$
$$= \epsilon_{\mathfrak{P}}\left(\sqrt{v_{\mathcal{V}_1}}\psi_{(\mathfrak{P})\mathcal{V}_1}\right)\longleftrightarrow$$
$$\longleftrightarrow\sum_{\mathcal{V}_2}\left[-\frac{i}{v_{\mathcal{V}_1}}\mathcal{D}_{\mathcal{V}_1,\mathcal{V}_2}\right]\psi_{(\mathfrak{P})\mathcal{V}_2} = \epsilon_{\mathfrak{P}}\psi_{(\mathfrak{P})\mathcal{V}_1} \tag{9}$$

is correct, and the set of eigenfunctions $\left\{\psi_{(\mathfrak{P})}\right\}$ forms a complete orthonormal basis in the metric (8). Let's expand the Dirac fields in this basis:

$$\psi_{\mathcal{V}} = \sum_{\mathfrak{P}}\eta_{\mathfrak{P}}\psi_{(\mathfrak{P})\mathcal{V}},\quad \psi^{\dagger}_{\mathcal{V}} = \sum_{\mathfrak{P}}\eta^{\dagger}_{\mathfrak{P}}\psi^{\dagger}_{(\mathfrak{P})\mathcal{V}}. \tag{10}$$

The new dynamic variables $\{\eta_{\mathfrak{P}},\,\eta^{\dagger}_{\mathfrak{P}}\}$ are Grassmann. The scalar product (8) in these variables is rewritten as

$$\langle\psi_1|\psi_2\rangle = \sum_{\mathfrak{P}}\eta^{\dagger}_{(1)\mathfrak{P}}\eta_{(2)\mathfrak{P}}. \tag{11}$$

It is important here that

$$\gamma^{5}i\mathcal{D}_{\mathcal{V}_1,\mathcal{V}_2} = -i\mathcal{D}_{\mathcal{V}_1,\mathcal{V}_2}\gamma^{5}. \tag{12}$$

The long-wavelength limit of the theory is straightforward. To do this one must believe the quantities $A_{\mathcal{W}ij}$ and $e^{a}_{\mathcal{W}ij}$ as the smooth 1-forms

$$A_{\mathcal{W}ij}\to A_a(x)dx^a,\quad e^{a}_{\mathcal{W}ij}\to dx^a$$

taking the small values $A_{\mathcal{W}ij}$ and $e^{a}_{\mathcal{W}ij}$ on the vector $e^{a}_{\mathcal{W}ij}$, and substitute the smooth Dirac field $\psi(x)$ taking the value $\psi_{\mathcal{V}}$ on the vertex $a_{\mathcal{V}}$ for the set of spinors $\psi_{\mathcal{V}}$. As a result the action (4), the scalar product (8), and the eigenvalue problem (9) transform to the well known expressions and equation:

$$\mathfrak{A}_{\psi} = \int\left[-i\psi^{\dagger}\gamma^{a}\nabla_{a}\psi\right]dx^1\wedge dx^2\wedge dx^3\wedge dx^4,$$
$$\nabla_a = \partial_a + ieA_a, \tag{13}$$

$$\langle\psi_1|\psi_2\rangle = \int\psi^{\dagger}_{(1)}(x)\psi_{(2)}(x)d^{(4)}x, \tag{14}$$

$$-i\gamma^{a}\nabla_{a}\psi_{(\mathfrak{P})}(x) = \epsilon_{\mathfrak{P}}\psi_{(\mathfrak{P})}(x). \tag{15}$$

3. The partition function of the fermion system as the functional of the quantities $\{e^{a}_{\mathcal{W}ij}\}$ and $\{A_{\mathcal{W}ij}\}$ is given by integral

$$Z\{e^{a}_{\mathcal{W}ij},\,A_{\mathcal{W}ij}\} = \int\left(\mathcal{D}\psi^{\dagger}\mathcal{D}\psi\right)\exp\mathfrak{A}_{\psi}. \tag{16}$$

Here the fermion functional measure is defined according to

$$\left(\mathcal{D}\psi^{\dagger}\mathcal{D}\psi\right)\equiv\prod_{\mathcal{V}}d\psi^{\dagger}_{\mathcal{V}}d\psi_{\mathcal{V}}F\{e^{a}_{\mathcal{W}ij}\}, \tag{17}$$

where

$$d\psi_{\mathcal{V}} = \prod_{\varkappa=1}\prod_{s=1}^{4}d\psi_{\mathcal{V}\varkappa s},\quad d\psi^{\dagger}_{\mathcal{V}} = \prod_{\varkappa=1}\prod_{s=1}^{4}d\psi^{\dagger}_{\mathcal{V}\varkappa s}, \tag{18}$$

and the index $\varkappa$ enumerates the components of the gauge representation, while the index $s = 1, 2, 3, 4$ is the Dirac one. The functional $F\{e^a_{\mathcal{W}ij}\}$ in (17) can be calculated easily with the help of the metric (8), but it is not interesting here. The scalar product (11) in Grassmann variables $\{\eta_{\mathfrak{P}}, \eta^\dagger_{\mathfrak{P}}\}$ permits to rewrite the measure (17) as below:

$$\left(\mathcal{D}\psi^\dagger\mathcal{D}\psi\right) = \prod_{\mathfrak{P}} d\eta^\dagger_{\mathfrak{P}} d\eta_{\mathfrak{P}}. \qquad (19)$$

Let's study the chiral transformation of the Dirac field

$$\psi_{\mathcal{V}} \to \exp\left(i\alpha_{\mathcal{V}}\gamma^5\right)\psi_{\mathcal{V}}, \quad \psi^\dagger_{\mathcal{V}} \to \psi^\dagger_{\mathcal{V}} \exp\left(i\alpha_{\mathcal{V}}\gamma^5\right). \quad (20)$$

Obviously, the measure (17) is invariant under the transformation (20). Moreover, even the factors $\left(\prod_{s=1}^4 d\psi_{\mathcal{V}\varkappa s}\right)$ and $\left(\prod_{s=1}^4 d\psi^\dagger_{\mathcal{V}\varkappa s}\right)$ of the measure (17) each are invariant since the matrix γ^5 is traceless. It follows from here that the measure in right-hand side of Eq. (19) is also invariant under the chiral transformation and the corresponding Jacobian $J = 1$. The last statement permits to extract some interesting information.

Suppose the chiral transformation is infinitesimal: $\alpha_{\mathcal{V}} \to 0$. Then from the linearized transformations of the Dirac field (20) we obtain also linearized transformations for the variables $\{\eta_{\mathfrak{P}}, \eta^\dagger_{\mathfrak{P}}\}$:

$$\begin{aligned} \eta_{\mathfrak{P}} &\to \eta_{\mathfrak{P}} + i\sum_{\mathfrak{Q}} \eta_{\mathfrak{Q}} \sum_{\mathcal{V}} \alpha_{\mathcal{V}} \upsilon_{\mathcal{V}} \psi^\dagger_{\mathfrak{P}\mathcal{V}}\gamma^5\psi_{\mathfrak{Q}\mathcal{V}}, \\ \eta^\dagger_{\mathfrak{P}} &\to \eta^\dagger_{\mathfrak{P}} + i\sum_{\mathfrak{Q}} \eta^\dagger_{\mathfrak{Q}} \sum_{\mathcal{V}} \alpha_{\mathcal{V}} \upsilon_{\mathcal{V}} \psi^\dagger_{\mathfrak{Q}\mathcal{V}}\gamma^5\psi_{\mathfrak{P}\mathcal{V}}. \end{aligned} \qquad (21)$$

The Jacobian of this transformation is equal to

$$J = \left(1 + 2i\sum_{\mathcal{V}} \alpha_{\mathcal{V}}\upsilon_{\mathcal{V}} \sum_{\mathfrak{P}} \psi^\dagger_{\mathfrak{P}\mathcal{V}}\gamma^5\psi_{\mathfrak{P}\mathcal{V}}\right).$$

On the other hand, as was stated before, $J = 1$. Therefore, since the quantities $\alpha_{\mathcal{V}}$ are arbitrary at each vertex, we have

$$\sum_{\mathfrak{P}} \psi^\dagger_{\mathfrak{P}\mathcal{V}}\gamma^5\psi_{\mathfrak{P}\mathcal{V}} = 0. \qquad (22)$$

For the following analysis it is necessary to decompose the sum (22) into infrared or long-wavelength and the rest ultraviolet parts. Firstly let's consider the infrared part. One must introduce the following scales: the gauge field wavelength order $\sim \lambda$; the scale of ultraviolet cutoff of the long-wavelength sector Λ; the lattice scale $l_P \sim \left|e^a_{\mathcal{W}ij}\right|$. By assumption

$$\lambda^{-1} \ll \Lambda \ll l_P^{-1}. \qquad (23)$$

The value of the long-wavelength part of the sum (22) is well known:

$$\begin{aligned} \sum_{|\epsilon_{\mathfrak{P}}|<\Lambda} \psi^\dagger_{\mathfrak{P}}(x)\gamma^5\psi_{\mathfrak{P}}(x) &= -\frac{e^2}{32\pi^2}\varepsilon^{abcd}\mathrm{tr}\left\{F_{ab}(x)F_{cd}(x)\right\}, \\ F_{ab} &= \partial_a A_b - \partial_b A_a + ie\left[A_a,\, A_b\right]. \end{aligned} \qquad (24)$$

It is well known that the value (24) is a one-half of the axial vector anomaly. Here the expression for the anomaly is extracted from the fermion measure (19). This method was suggested by Vergeles [9] and Fujikawa [10].

Note that the value of the sum in (24) does not depend on the cutoff parameter Λ if it is enclosed in a range of values (23). This fact in turn means that

$$\sum_{\Lambda_1<|\epsilon_{\mathfrak{P}}|<\Lambda_2} \psi^\dagger_{\mathfrak{P}}(x)\gamma^5\psi_{\mathfrak{P}}(x) = 0, \quad \lambda^{-1} \ll \Lambda_1 < \Lambda_2 \ll l_P^{-1}.$$

It is clear from here that the decomposition of the sum in (22) into long-wavelength and ultraviolet parts is well defined.

The comparison of Eqs. (22) and (24) leads to the following equality:

$$\sum_{|\epsilon_{\mathfrak{P}}|>\Lambda} \psi^\dagger_{\mathfrak{P}\mathcal{V}}\gamma^5\psi_{\mathfrak{P}\mathcal{V}} = \frac{e^2}{32\pi^2}\varepsilon^{abcd}\mathrm{tr}\left\{F_{ab}(x)F_{cd}(x)\right\}. \qquad (25)$$

The sums in (24) and (25) do not depend on the cutoff parameter Λ and they become saturated at $|\epsilon_{\mathfrak{P}}| \sim \lambda^{-1}$ and $|\epsilon_{\mathfrak{P}}| \sim l_P^{-1}$, correspondingly.

Now let us integrate the equalities (24) and (25) over the space:

$$\begin{aligned} &\sum_{|\epsilon_{\mathfrak{P}}|<\Lambda} \int d^{(4)}x\psi^\dagger_{\mathfrak{P}}(x)\gamma^5\psi_{\mathfrak{P}}(x) = \\ &= -\frac{e^2}{32\pi^2}\varepsilon^{abcd}\int d^{(4)}x\mathrm{tr}\left\{F_{ab}(x)F_{cd}(x)\right\} = q, \end{aligned} \qquad (26)$$

$$\begin{aligned} &\sum_{|\epsilon_{\mathfrak{P}}|>\Lambda} \frac{1}{5}\sum_{\mathcal{V}} \upsilon_{\mathcal{V}}\psi^\dagger_{\mathfrak{P}\mathcal{V}}\gamma^5\psi_{\mathfrak{P}\mathcal{V}} = \\ &= \frac{e^2}{32\pi^2}\varepsilon^{abcd}\int d^{(4)}x\mathrm{tr}\left\{F_{ab}(x)F_{cd}(x)\right\} = -q. \end{aligned} \qquad (27)$$

Here $q = 0, \pm 1, \ldots$ is the topological charge of the gauge field instanton. In consequence of Eq. (12) only zero modes give the contributions to the sums over $\mathfrak{P}$ in the left-hand sides of Eqs. (26) and (27). Using Eq. (26) and the identity $\gamma^5 \equiv \left[(1+\gamma^5)/2 - (1-\gamma^5)/2\right]$ we obtain the relatin

$$n_+ - n_- = q, \qquad (28)$$

where n_+ (n_-) is the number of the right (left) zero modes of Eq. (15). This relation represents the famous Atiyah–Singer index theorem [11].

With the help of Eq. (27) we obtain by the same procedure the relation

$$n_+^{\mathcal{I}} - n_-^{\mathcal{I}} = -q. \tag{29}$$

Here $n_+^{\mathcal{I}}$ ($n_-^{\mathcal{I}}$) is the number of the right (left) *irregular* zero modes of Eq. (9). The difference between the usual and irregular modes is as follows: For the usual modes and adjacent vertices $a_{\mathcal{W}i}$ and $a_{\mathcal{W}j}$ we have

$$\left|\psi_{(\mathfrak{P})\mathcal{W}i} - \psi_{(\mathfrak{P})\mathcal{W}j}\right| \sim l_P\, \epsilon_{\mathfrak{P}} \left|\psi_{(\mathfrak{P})\mathcal{W}j}\right| \to 0. \tag{30}$$

By definition, the irregular modes cannot satisfy the estimation (30), but they satisfy the estimation

$$\left|\psi^{\mathcal{I}}_{(\mathfrak{P})\mathcal{W}i} - \psi^{\mathcal{I}}_{(\mathfrak{P})\mathcal{W}j}\right| \sim \left|\psi^{\mathcal{I}}_{(\mathfrak{P})\mathcal{W}i}\right| \tag{31}$$

at least at a part of vertices. Thus, the usual and irregular modes are well separated not only by the energy $\epsilon_{\mathfrak{P}}$ but also by the "momentum".

The relations (25) and (29) are the main results of the paper.

4. Denote by $\psi^{\mathcal{I}}_{(\pm 0\,\xi)\mathcal{V}}$ the left or right irregular zero mode of Eq. (9):

$$\sum_{\mathcal{V}_2} \left[-\frac{i}{v_{\mathcal{V}_1}} \mathcal{D}_{\mathcal{V}_1,\mathcal{V}_2}\right] \psi^{\mathcal{I}}_{(\pm 0\,\xi)\mathcal{V}_2} = 0. \tag{32}$$

The index ξ enumerates the zero modes.

Now let's denote by $\left[-\,(i/v_{\mathcal{V}_1})\,\mathcal{D}^{(\text{free})}_{\mathcal{V}_1,\mathcal{V}_2}\right]$ the free lattice Dirac operator (see the left-hand sides of Eqs. (32) and (9)) in which $U_{\mathcal{W}mi} = 1$ is used instead of $U_{\mathcal{W}mi} = \exp\left(ieA^{(\text{inst})}_{\mathcal{W}mi}\right)$. In other words, the free Dirac operator is obtained from the general one by the gauge field elimination. It is easy to obtain the following estimation:

$$\sum_{\mathcal{V}_2} \left[-\frac{i}{v_{\mathcal{V}_1}} \mathcal{D}^{(\text{free})}_{\mathcal{V}_1,\mathcal{V}_2}\right] \psi^{\mathcal{I}}_{(\pm 0\,\xi)\mathcal{V}_2} = \mathrm{O}\left(\frac{e}{\rho}\left|\psi^{\mathcal{I}}_{(\pm 0\,\xi)\mathcal{V}_1}\right|\right). \tag{33}$$

Here ρ is the scale of the instanton field $A^{(\text{inst})}_{\mathcal{W}mi}$. The proof of (33) is based on the estimations

$$A^{(\text{inst})}_{\mathcal{W}mi} \sim (l_P/\rho) \ll 1,$$

$$1 \approx \exp\left(ieA^{(\text{inst})}_{\mathcal{W}mi}\right) - ieA^{(\text{inst})}_{\mathcal{W}mi} = U_{\mathcal{W}mi} + \mathrm{O}\left(\frac{e\,l_P}{\rho}\right),$$

and the fact that the lattice Dirac operator is linear in the $U_{\mathcal{W}mi}$. Therefore

$$\mathcal{D}^{(\text{free})}_{\mathcal{V}_1,\mathcal{V}_2} = \mathcal{D}_{\mathcal{V}_1,\mathcal{V}_2} + \mathrm{O}\left(\frac{e\,l_P^4}{\rho}\right).$$

Thus the estimation (33) follows from Eq. (32).

The estimation (33) allows to do the final conclusion: the Wilson fermion doubling phenomenon on irregular lattices does exist. Otherwise, the energy gap of the order of $\epsilon^{\mathcal{I}}_{\mathfrak{P}} \sim 1/l_P$ would be expected to take place in the sector of irregular modes of the free Dirac operator. In any case the expansion of each spinor wave function $\psi^{\mathcal{I}}_{(\pm 0\,\xi)\mathcal{V}}$ in terms of the eigenfunctions of the free Dirac operator contains the irregular modes of the operator. Thus, the additional contribution of the order of (c/l_P) would be in the right-hand side of the estimation (33), the number $c \neq 0$. But the right hand side of the estimation (33) does not depend on the lattice parameter l_P. Thus there are the low energy irregular Dirac modes. The classification of the modes should be a subject of future scientific research.

Finally, it is worth to note, that the suggested approach is valid also for the regular lattices.

This work was supported by SS-3139.2014.2.

1. K. Wilson, Erice lectures notes **CLNS-321** (1975).
2. J. Kogut and L. Susskind, Phys. Rev. D **11**, 393 (1975).
3. L. Susskind, Phys. Rev. D **16**, 3031 (1977).
4. M. Luscher, arXiv:hep-th/0102028.
5. H. Nielsen and M. Ninomiya, Nucl. Phys. B **185**, 20 (1981).
6. H. Nielsen and M. Ninomiya, Nucl. Phys. B **193**, 173 (1981).
7. S. N. Vergeles, Nucl. Phys. B **735**, 172 (2006).
8. S. N. Vergeles, JETP **106**, 46 (2008).
9. S. N. Vergeles (unpublished), quoted in: A. A. Migdal, Phys. Lett. B **81**, 37 (1979).
10. K. Fujikawa, Phys. Rev. Lett. **42**, 1195 (1979).
11. M. F. Atiyah and I. M. Singer, Proc. Nat. Acad. Sci. **81**, 2597 (1984).