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Flat coordinates of topological CFT and solutions of Gauss–Manin system

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It was shown many years ago by Dijkgraaf, Velinde, Verlinde for $2d$ Topological Conformal Field Theory (TCFT) and more recently for the non-critical String theory that some models of these two types can be solved using their connection with the special case of Frobenius Manifolds (FM) so called Saito Frobenius manifolds connected with a deformed singularity. The crucial point for obtaining an explicit expression for the correlators is finding *the flat coordinates* of SFMs as functions of the parameters of the deformed singularity. We suggest a direct way to find the flat coordinates, using the integral representation for the solutions of Gauss–Manin system connected with the corresponding SFM for the singularity.

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It is known that space-time supersymmetry in string theory after the compactification is achieved due to $N=2$ superconformal symmetry on the worldsheet.

In [1] it was shown that the twisted restriction to the chiral states $N=2$ models gives a Topological Conformal Field Theory (TCFT). This fact is a main reason why TCFT is important for string theory.

In [2] a relation was established between chiral rings of Kazama–Suzuki $SU(N)_k/[SU(N-1)_{k+1}\times U(1)]$ $N=2$ models and the theory of singularities.

It is of great interest to solve the mentioned theories explicitly. In [3] the important series of minimal TCFT or $SU(2)_k/U(1)$ theory was solved. This is the first exact solution of TCFT models. It was accomplished due to their connection with the of Frobenius manifold structure.

Frobenius manifold M is a Riemann manifold with coordinates $t^0, t^2, \ldots, t^{\dim M-1}$, a flat metric $g_{\alpha\beta}$ and a tensor $C^{\gamma}_{\alpha\beta}$ whose components are the structure constants of an associative Frobenius algebra $e_\alpha e_\beta = C^{\gamma}_{\alpha\beta}e_\gamma$ which is identified with the tangent space of M $\frac{\partial}{\partial t^\alpha}\to \to e_\alpha$.

The Riemann and Frobenius structures are asumed to be consistent as follows: $g_{\alpha\beta}=(e_\alpha,e_\beta)$ is invariant, which means $(e_\alpha e_\beta, e_\gamma)=(e_\alpha, e_\beta e_\gamma)$ and $\nabla_\alpha C^{\mu}_{\beta\gamma} = \nabla_\beta C^{\mu}_{\alpha\gamma}$ where ∇_α is the Levi–Civita connection for the flat metric $g_{\alpha\beta}(t)$.

Let f be a quasihomogenous superpotential of an isolated singularity, $e_\alpha, \alpha=0,...,\dim M$ are elements of the basis of its Jacobi Ring and F – its versal deformation

$$F = f + \sum_{\alpha=0}^{\dim M-1} t^\alpha e_\alpha. \qquad (1)$$

It was proposed by K. Saito [4], there exists a form of the top degree ζ, so called *primitive form*, such that the metric, defined by means ζ as

$$g_{\alpha\beta} = \mathrm{Res}\,\frac{e_\alpha e_\beta \zeta}{\prod \partial_i F}, \qquad (2)$$

is flat and together with the structure constants

$$C_{\alpha\beta\gamma} = \mathrm{Res}\,\frac{e_\alpha e_\beta e_\gamma \zeta}{\prod \partial_i F} \qquad (3)$$

provide a Frobenius manifold structure. In particular, primitive forms for simple singularities are, up to a constant factor, given by the standard volume form dx.

It was shown in [1] the Topolical CFT (TCFT) can be obtained from $N=2$ Supersymmetric CFT (SCFT) by twisting of the energy-momentum tensor

$$T_{\mathrm{top}}(z) = T^{N=2}(z) + \frac{1}{2}\partial J^{N=2}$$

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and restricting the space of local fields of $N=2$ SCFT model to its subspace consisting of the Chiral fields and their superpartners.

The symmetry of the TCFT is generated by operators L_n, Q_n, G_n, J_n, which satisfy to usual $N=2$ commutation relations [3].

These relations include

$$L_n = \{Q, G_n\}, \quad Q^2 = 0, \quad Q = Q_0,$$

which is exactly the definition of TCFT [3, 1].

In the case of twisting of $N=2$ Minimal models, connected with the coset $SU(2)_k/U(1)$, we have $k+1$ chiral fields Φ_l, $l = 1, ..., k$.

Besides Φ_l the correlators in TCFT include also their superpartners $\tilde{\Phi}_l$

$$\tilde{\Phi}_l = G_{-1}\bar{G}_{-1}\Phi_l.$$

From the commutation relations above it follows that

$$Q\tilde{\Phi}_l = \partial\bar{G}_{-1}\Phi_l \quad \Rightarrow \quad Q\int\tilde{\Phi}_l = 0.$$

All the information about the correlators of the theory is contained in the perturbed correlators [3]

$$F^{(s)}_{i_1 i_2 \ldots i_n} = \langle \Phi_{i_1}(z_1)\ldots\Phi_{i_n}(z_n) e^{\sum s^l \int \tilde{\Phi}_l} \rangle,$$

which do not depend of the coordinates z_i since

$$\partial\Phi_l = QG_{-1}\Phi_l \quad \Rightarrow \partial_i\langle\Phi_{i_1}\ldots\Phi_{i_n}\rangle = 0.$$

It was shown in [3] that all the information about $F_{i_1 i_2 \ldots i_n}(s)$ is known, if we know two of them

$$C_{ijk}(s) = \langle \Phi_i\Phi_j\Phi_k e^{\sum s^l \int \tilde{\Phi}_l} \rangle,$$

$$\eta_{ij} = C_{ij0} = \langle \Phi_i\Phi_j e^{\sum s^l \int \tilde{\Phi}_l} \rangle.$$

η_{ij} and $C_{ijk}(s)$ satisfy, as was proved in [3], the following constraints.

η_{ij} does not depend on s^l, C_{ijk} – symmetric tensor and

$$\partial_m C_{ijk} = \partial_i C_{mij},$$

$$C^l_{ij} C^n_{lk} = C^n_{il} C^l_{jk}.$$

These relations show that we obtain Frobenius manifold structure [5].

$C^l_{ij}(s)$ are structure constants of some Ring R which is a deformation of the Chiral Ring of the Minimal model.

The coupling constants s^i in the definition of the perturbed correlators are the flat coordinates on the Frobenius manifold and η_{ij} is a flat metric in these coordinates.

Also it was proved in [3] that this perturbed ring R in the case of $SU(2)_k/U(1)$ Minimal models coinside with the ring of the versal deformation of the singularity $W_0 = x^{k+2}$

$$\mathbb{C}[x]/\partial_x W, \tag{4}$$

where the perturbed singularity

$$W = x^{k+2} + \sum_{i=0}^{k} t_i x^i. \tag{5}$$

The coupling constants $s_i = s_i(t_1, ..., t_k)$ are functions of the perturbation parameters t_i. Thus they are nothing but *the Flat coordinates* for the Frobenius manifold of the ring for the perturbed singularity $\mathbb{C}[x]/\partial_x W$, where

$$W = x^{k+2} + \sum_{i=0}^{k} t_i x^i. \tag{6}$$

This implies that

$$C_{ijk}(s) = \frac{\partial t^a}{\partial s^i}\frac{\partial t^b}{\partial s^j}\frac{\partial t^c}{\partial s^k}\mathrm{Res}_{x=\infty}\frac{x^{a+b+c}dx}{\partial_x(x^{k+2} + \sum t_i x^i)}. \tag{7}$$

In fact this formula gives the exact solution for $SU(2)_k/U(1)$ minimal models of topological CFT.

Thus to get the explicit expressions for the correlation functions of the minimal models we have to know the Flat coordinates s^i as functions of the deformation parameters t^i.

Consider the case of a simple singularity defined by a quasi-homogeneous holomorphic function $f(x) \in \mathbb{C}[x_i]$. That the function f is quasi-homogeneous means that there are weights ν_i such that

$$f(\lambda^{\nu_i} x_i) = \lambda f(x_i). \tag{8}$$

Milnor ring is defined by

$$R_0 = \mathbb{C}[x]/(\partial_1 f, ..., \partial_n f). \tag{9}$$

We pick up a quasi-homogeneous basis e_μ with weights $[e_\mu]$ such that $e_0 = 1$ of R_0 and consider a deformation

$$F(x_i, t_\alpha) = f - t_0 e_0 - t_\alpha e_\alpha, \tag{10}$$

$$R = \mathbb{C}[x]/(\partial_1 F, ..., \partial_n F). \tag{11}$$

Taking the weights ϵ_μ of the parameters t_μ to be equal $1 - [e_\mu]$ we ensure that F is also quasi-homogeneous

$$F(\lambda^{\nu_i}x_i, \lambda^{\epsilon_\mu}t_\mu) = \lambda F(x_i, t_\mu). \quad (12)$$

Define tensors C, A, and B as

$$e_\alpha e_\beta = C^\gamma_{\alpha\beta} e_\gamma + A^i_{\alpha\beta}\partial_i F, \quad (13)$$
$$\textstyle\sum_i \partial_i A^i_{\alpha\beta} = -B^\gamma_{\alpha\beta} e_\gamma. \quad (14)$$

Since the singularity is simple, or all weights $e_\mu < 1$, we don't need to add the term proportional to the derivatives in the second equation.

Define

$$\Psi^{(n)} = \int_{C_n} \frac{dx}{F}, \quad (15)$$

where the contours are defined to be eigen functions with the same monodromy as t^μ and

$$\Psi^{(\mu)}_\alpha = \partial^{-1}_{t_0}\partial_{t_\alpha}\Psi^{(\mu)}, \quad (16)$$

which satisfies the so-called Gauss–Manin system of equations:

$$\partial_{t_\alpha}\Psi^{(n)}_\beta = C^\gamma_{\alpha\beta}\partial_{t_0}\Psi^{(n)}_\gamma + B^\gamma_{\alpha\beta}\Psi^{(n)}_\gamma. \quad (17)$$

Note that from the definition of tensors C, A, and B follows that they do not depend on t_0.

It is convenient to rewrite the Gauss–Manin system as a matrix differential equation for the column $\Psi^{(n)}$ whose components are $\Psi^{(n)}_\gamma$

$$(\partial_\alpha - C_\alpha\partial_0 - B_\alpha)\,\Psi^{(n)} = 0, \quad (18)$$

where C_α and B_α are matrices whose entries are equal to $C^\gamma_{\alpha\beta}$ and $B^\gamma_{\alpha\beta}$.

Since the matrices C_α and B_α do not depend on t_0 we obtain the following compatibility equations for the GM system

$$[C_\alpha, C_\beta] = 0, \quad (19)$$

$$\partial_\alpha C_\beta - \partial_\beta C_\alpha - [C_\alpha, B_\beta] + [C_\beta, B_\alpha] = 0, \quad (20)$$

$$\partial_\alpha B_\beta - \partial_\beta B_\alpha - [B_\alpha, B_\beta] = 0. \quad (21)$$

Eq. (34) for $B^\gamma_{\alpha\beta}$ together with the relation

$$B^\gamma_{\alpha\beta} = B^\gamma_{\beta\alpha} \quad (22)$$

show that $B^\gamma_{\alpha\beta}$ can be considered as Kristoffel coefficients for Levi–Civita connections of some flat metric.

Taking into account this fact we conclude that Eqs. (32)–(34) are nothing but Dubrovin axioms for a Frobenius manifold structure.

Many checks support the *Conjecture* that $B^\gamma_{\alpha\beta}$ is identical to the Kristoffel coefficients for Levi–Civita connections of metric

$$g_{\alpha\beta} = \mathrm{Res}\,\frac{e_\alpha e_\beta}{\partial_x F(x;t)}. \quad (23)$$

The integral representation for the solutions of Gauss–Manin system can be used for getting the explicit expressions of the flat coordinates s^μ through the deformation papameters t_α.

To do this it is convenient to choose a special fundamental system of the solutions of Gauss–Manin system. Namely, we will use the solutions which are the eigenfunctions under the monodromy group of transformations, whose action is defined as $x_i \to \exp(i\cdot 2\pi\nu_i)x_i$, $t_0 \to t_0$ and $t_\mu \to \exp(i\cdot 2\pi\epsilon_\mu)t_\mu$.

The Gauss–Manin system is invariant under such a transformation. Therefore we can choose our fundamental system of its solution Ψ^μ such that

$$\Psi^\mu_0[t_0, \exp(2\pi i\epsilon_\beta)t_\beta] = \exp(2\pi i\epsilon_\mu)\Psi^\mu_0(t_0, t_\beta) \quad (24)$$

and

$$\partial_0\Psi^\mu_\alpha(t_0, t_\sigma) = \partial_\alpha\Psi^\mu_0(t_0, t_\sigma). \quad (25)$$

From the first of these equations it follows that

$$\Psi^\mu_0(t_0, t_\beta) = t_0^{\epsilon_\mu} G\left(\frac{t_\beta}{t_0^{\epsilon_\beta}}\right). \quad (26)$$

Then if the combinations $\frac{t_\beta}{t_0^{\epsilon_\beta}}$ are small we have an expansion

$$\Psi^\mu_0(t_0, t_\sigma) = \sum_{N=0} \frac{U^\mu(t_\beta, N)}{t_0^{\epsilon-\epsilon_\mu+N}}, \quad (27)$$

where

$$U^\mu(t_\beta, N) = \sum_{m_i:\,\sum\epsilon_i m_i = \epsilon_\mu + N} C^\mu(m_\beta)\left(\prod\frac{t_i^{m_i}}{m_i!}\right). \quad (28)$$

Similary for $\Psi^\mu_\alpha(t_0, t_\sigma)$ we obtain an expansion

$$\Psi^\mu_\alpha(t_0, t_\sigma) = \sum_{N=0} \frac{U^\mu_\alpha(t_\beta, N)}{t_0^{\epsilon+\epsilon_\alpha-\epsilon_\mu+N}} \quad (29)$$

and from Eq. (38) it follows that

$$U^\mu_\alpha(t_\sigma, N=0) := U^\mu_\alpha(t_\sigma) = \partial_\alpha U^\mu_0(t_\sigma, N=0). \quad (30)$$

Structure constants $C^\gamma_{\alpha\beta}$ and $B^\gamma_{\alpha\beta}$ do not depend on t_0. From this fact and Eqs. (29) and (42) it follows that

$$(\partial_\alpha - B_\alpha)U^\mu_\beta(t_\sigma) = 0. \quad (31)$$

This means that

$$U^{\mu}_{\alpha}(t_\sigma) = \frac{\partial s^{\mu}}{\partial t^{\alpha}}, \tag{32}$$

where the functions $s^{\alpha}(t_\beta)$ are the Flat coordinates.

Comparing Eqs. (45) and (43) we arrive to the conclusion that the flat coordinates coincide with the leading terms for $\Psi^{\mu}_{0}(t_0, t_\sigma)$ in the expression (40). That is

$$s^{\mu}(t_\sigma) = U^{\mu}_{0}(t_\sigma). \tag{33}$$

Performing change of coordinates $t^{\alpha} \to s^{\alpha}$ a direct computation shows that

$$B^{\gamma}_{\alpha\beta} \to \tilde{B}^{\gamma}_{\alpha\beta} = 0, \tag{34}$$

$$C^{\gamma}_{\alpha\beta} \to \tilde{C}^{\gamma}_{\alpha\beta}. \tag{35}$$

In the new coordinates we have

$$\partial_\alpha \tilde{C}_{\beta\gamma\delta} = \partial_\beta \tilde{C}_{\alpha\gamma\delta}. \tag{36}$$

Taking into account that $\tilde{C}_{\beta\gamma\delta}$ do not depend on s^0 we obtain

$$\partial_\alpha \tilde{C}_{0\beta\gamma} = \partial_0 \tilde{C}_{\alpha\beta\gamma} = 0. \tag{37}$$

Taking into account that $\tilde{C}_{0\alpha\beta} = \tilde{g}_{\alpha\beta}$ we obtain that the metric $g_{\alpha\beta}$ defined in (36) in the new coordinates is constant.

So it confirms our conjecture that $B^{\gamma}_{\alpha\beta}$ are Kristoffel symbols for the Levi–Civita connection of $g_{\alpha\beta}$.

In this part we will show how to use the integral representation (27) for $\Psi^{\mu}_{0}(t_0, t_\sigma)$ for obtaining the explicit expression for the flat coordinates.

We will consider the example of the versal deformation of the one-dimensional A_p-singularity

$$F = x^{p+1} - t_0 + \sum_{1 \le i \le p-1} t_i x^i. \tag{38}$$

Let us choose the initial coutours C_n in the definition $\Psi^{(n)}_{0}$ as shown on Fig. 1

$$\Psi^{(n)}_{0} = \int_{C_n} \frac{dx}{F}. \tag{39}$$

In this case, the integral is taken around the pole in

$$x_n = (t_0)^{1/(p+1)} \exp[2\pi i n/(p+1)] \tag{40}$$

if t_i are small for $i > 0$.

From their definition $\Psi^{(n)}_{0}(t_0, t_\beta)$ satisfy the relation

$$\Psi^{(n)}_{0}(t_0, \omega^{(p+1-i)} t_i) = \bar{\omega} \Psi^{(n+1)}_{0}(t_0, t_i), \tag{41}$$

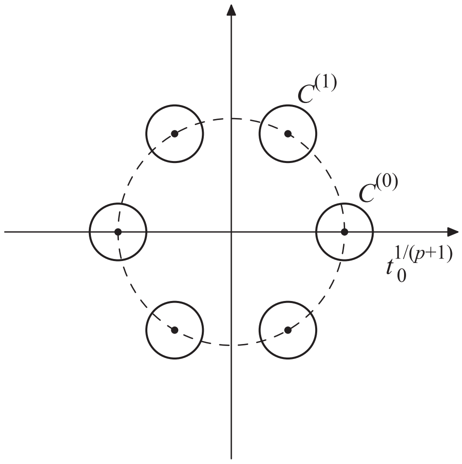

Fig. 1. Initial contours

where $\omega = \exp \frac{2\pi i}{p+1}$.

For our aim it is convenient to choose instead of $\Psi^{(n)}_{0}(t_0, t_i)$ their linear combination

$$\Psi^{\mu}_{0}(t_0, t_i) = \sum_{n} \bar{\omega}^{\mu n} \Psi^{(n)}_{0}(t_0, t_i) \tag{42}$$

which are eigenfunctions of the monodromic transformation

$$\Psi^{\mu}_{0}(t_0, \omega^{(p+1-i)} t_i) = \omega^{\mu} \Psi^{\mu}_{0}(t_0, t_i). \tag{43}$$

As was explaned above, when the combinations $\frac{t_i}{t_0^{\epsilon_i}}$, where $\epsilon_i = \frac{p+1-i}{p+1}$, are small, we have the expansion

$$\Psi^{\mu}_{0}(t_0, t_i) = \sum_{N=0} \frac{U^{\mu}(t_i, N)}{t_0^{\epsilon - \epsilon_m u + N}}, \tag{44}$$

where

$$U^{\mu}(t_\beta, N) = \sum_{m_i:\, \sum \epsilon_i m_i = \epsilon_\mu + N} C^{\mu}(m_\beta) \left(\prod \frac{t_i^{m_i}}{m_i!} \right). \tag{45}$$

For computation of $\Psi^{\mu}_{0}(t_0, t_i)$ we will use the integral representation (52) for $\Psi^{(n)}_{0}(t_0, t_i)$.

Indeed we don't need to compute the integrals along all contours $C^{(n)}$ and take their linear combination defined in (55).

The weighted summing in (55) plays only a role of the projector of each of the integrals to the part which correspond the eigenfunction with a definite monodromic behavior.

Therefore we will concentrate on computing one of the integrals, namely, along $C^{(0)}$ and leave in it only those terms of its serias which possesses the needed monodromic behaviour.

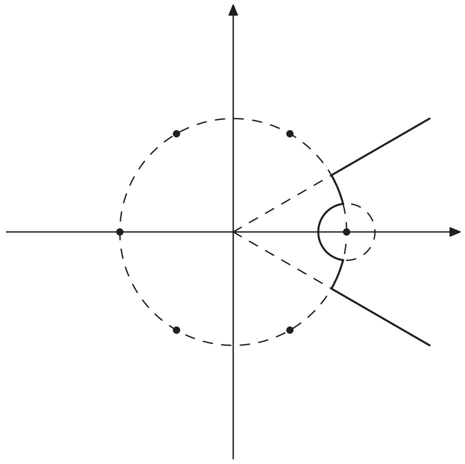

Fig. 2. Deformation of the contour $C^{(0)} \to \tilde{C}^{(0)}$

Deform the contour $C^{(0)} \to \tilde{C}^{(0)}$ as shown on the Fig. 2.

This deformation permits us to perform an integral representation for $1/F$ since

$$f(x) = x^{p+1} - t_0 \tag{46}$$

has negative real part on the contour.

It follows that we can use the integral representation:

$$F^{-1} = \int_0^\infty dy e^{Fy}. \tag{47}$$

Assuming that the variables $t_{\geq 1}$ are small we perform the expansion over t_i with $i \geq 1$

$$\Psi_0^{(0)} = \int_{\tilde{C}^{(0)}} dx \int_0^\infty dy e^{y(x^{p+1} - t_0 + \sum t_i x^i)} = = \int_{\tilde{C}^{(0)}} dx \int_0^\infty dy e^{yx^{p+1}} e^{-t_0 y} \times \times \sum_{m_i} \left(\prod \frac{t_i^{m_i}}{m_i!} \right) x^{\sum i m_i} y^{\sum m_i}. \tag{48}$$

Changing the order of integration and changing of variables

$$x \to y^{-1/(p+1)} x \tag{49}$$

we obtain

$$\Psi_0^{(0)} = \int dy y^{\sum m_i - \frac{\sum i m_i + 1}{p+1}} e^{-t_0 y} \times \times \left(\prod \frac{t_i^{m_i}}{m_i!} \right) \int_{\tilde{C}^{(0)}} dx e^{x^{p+1}} x^{\sum i m_i} = = \sum_{m_i} \frac{\Gamma\left(\sum m_i - \frac{\sum i m_i + 1}{p+1} + 1 \right)}{t_0^{\sum m_i - \frac{\sum i m_i + 1}{p+1} + 1}} \times \times \left(\prod \frac{t_i^{m_i}}{m_i!} \right) \int_{\tilde{C}^{(0)}} dx e^{x^{p+1}} x^{\sum i m_i}. \tag{50}$$

Deforming the contour $\tilde{C}^{(0)}$ to two rays shown on Fig. 2, we get that

$$\int_{\tilde{C}^{(0)}} dx e^{x^{p+1}} x^r = \frac{2\pi i}{p+1} \frac{1}{\Gamma\left(1 - \frac{r+1}{p+1}\right)}. \tag{51}$$

Introduce μ and N with $0 \leq \mu \leq p$ as

$$\sum (p+1-i) m_i = \mu + N(p+1). \tag{52}$$

Then we extract as has been explained above the expression for Ψ_0^μ

$$\Psi_0^\mu = \sum_{N \geq 0} \sum_{m_i : \sum (p+1-i) m_i = \mu + N(p+1)} \frac{2\pi i}{p+1} \times \times \left(\prod \frac{t_i^{m_i}}{m_i!} \right) \frac{1}{t_0^{\sum m_i - \frac{\sum i m_i + 1}{p+1} + 1}} \times \times \prod_{j=1}^{\sum m_i} \left(1 + N - \frac{1-\mu}{p+1} - j \right). \tag{53}$$

The flat coordinates coincide with the $N = 0$ term. So we obtain

$$s^\mu = \sum_{m_i : \sum (p+1-i) m_i = \mu} \left(\prod \frac{t_i^{m_i}}{m_i!} \right) \cdot (-1)^{\sum m_i} \times \times \prod_{j=0}^{\sum m_i - 1} \left(\frac{\mu+1}{p+1} + j \right). \tag{54}$$

It is possible to use our approach for solution the $SU(N)_k/[SU(N-1)_{k+1} \times U(1)]$ models [6], since they correspond to the isolated singularities what is the condition which ensures [4], the existance of Frobenius manifold structure on the space of its deformations.

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