Pis'ma v ZhETF, vol. 103, iss. 10, pp. 711 – 714

Brane realization of *q*-theory and the cosmological constant problem

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Submitted 20 April 2016

We discuss the cosmological constant problem using the properties of a freely-suspended two-dimensional condensed-matter film, i.e., an explicit realization of a 2D brane. The large contributions of vacuum fluctuations to the surface tension of this film are cancelled in equilibrium by the thermodynamic potential arising from the conservation law for particle number. In short, the surface tension of the film vanishes in equilibrium due to a thermodynamic identity. This 2D brane can be generalized to a 4D brane with gravity. For the 4D brane, the analogue of the 2D surface tension is the 4D cosmological constant, which is also nullified in full equilibrium. The 4D brane theory provides an alternative description of the phenomenological q-theory of the quantum vacuum. As for other realizations of the vacuum variable q, such as the 4-form field-strength realization, the main ingredient is the conservation law for the variable q, which makes the vacuum a self-sustained system. For a vacuum within this class, the nullification of the cosmological constant takes place automatically in equilibrium. Out of equilibrium, the cosmological constant can be as large as suggested by naive estimates based on the summation of zero-point energies. In this brane description, q-theory also corresponds to a generalization of unimodular gravity.

DOI: 10.7868/S0370274X16100040

1. Introduction. General relativity and the relativistic quantum field theory of the Standard Model are probably effective theories, as manifested by their ultraviolet divergences. Especially troublesome is the quartic divergence of the energy density from the vacuum fluctuations, which leads to the so-called cosmological constant problem (CCP) [1–3].

A particular mechanism for the nullification of the relevant vacuum energy density works for vacua which have the property of being *self-sustained media* [4–8]. A self-sustained medium has a definite macroscopic volume even in the absence of an environment. Two condensed-matter examples are a droplet of quantum liquid at zero temperature in empty space [9] and a freely-suspended film [10]. Here, we focus on the last example.

The Lorentz-invariant self-sustained medium relevant to the CCP is characterized by a relativistic scalar q. Distinct from a fundamental scalar ϕ, the vacuum variable q allows the medium to exist without external environment.

2. Fundamental vs. conserved scalars. In order to understand the difference between fundamental and conserved scalars, let us compare gravity with a fundamental scalar ϕ and gravity with a conserved scalar q obtained from a 4-form field strength F [11–18], with $F_{\kappa\lambda\mu\nu} \propto q\, \epsilon_{\kappa\lambda\mu\nu}$ and $q^2 \propto F_{\kappa\lambda\mu\nu}\, F^{\kappa\lambda\mu\nu}$. Let us consider these theories for the simplest possible arrangement: no explicit derivatives of ϕ and q, and no direct couplings of ϕ and q to the Ricci scalar R.

The action for gravity with a non-dynamic fundamental scalar ϕ is

$$I_1 = -\int_{\mathbb{R}^4} d^4x\, \sqrt{-g}\, \left(\frac{R}{16\pi G_N} + \epsilon_1(\phi) \right). \qquad (1)$$

The action for gravity with a three-form gauge field A is

$$I_2 = -\int_{\mathbb{R}^4} d^4x\, \sqrt{-g}\, \left(\frac{R}{16\pi G_N} + \epsilon_2(q) \right). \qquad (2)$$

Both functions $\epsilon_1(\phi)$ and $\epsilon_2(q)$ may include a genuine cosmological constant $\Lambda_{\rm bare}$.

Variation over $g^{\mu\nu}$ of the action I_j, for $j = 1$ or 2, gives the Einstein equation in standard form,

$$\frac{1}{8\pi G_N} \left(R_{\mu\nu} - \frac{1}{2}\, R\, g_{\mu\nu} \right) = \Lambda_j\, g_{\mu\nu}\,, \qquad (3)$$

but with different expressions for the vacuum energy density,

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$$\Lambda_1(\phi) = \epsilon_1(\phi), \tag{4a}$$

$$\Lambda_2(q) = \epsilon_2(q) - q\,\frac{d\epsilon_2(q)}{dq}. \tag{4b}$$

The calculation of the energy-momentum tensor gives an extra term in the $\Lambda(q)$ expression above, because the metric also enters the composite field q. Recall, in fact, the precise definitions $q^2 \equiv -(1/24)\,F_{\kappa\lambda\mu\nu}\,F^{\kappa\lambda\mu\nu}$ and $F_{\kappa\lambda\mu\nu} \equiv \nabla_{[\kappa}A_{\lambda\mu\nu]}$, with the three-form gauge field A, the covariant derivative ∇_κ, and a pair of square brackets around spacetime indices standing for complete anti-symmetrization.

The field equations for ϕ and q are also different,

$$\frac{d\epsilon_1(\phi)}{d\phi} = 0, \tag{5a}$$

$$\nabla_\nu\left(\frac{d\epsilon_2(q)}{dq}\right) = 0. \tag{5b}$$

The field equation for ϕ does not have the freedom to nullify the cosmological constant. The two requirements for nullification, $\epsilon_1(\phi) = 0$ and $d\epsilon_1(\phi)/d\phi = 0$, demand fine-tuning of the parameters entering the function $\epsilon_1(\phi)$, cf. Refs. [1, 2].

In contrast, the solution of the field equation for q is given by

$$\frac{d\epsilon_2(q)}{dq} = \mu, \tag{6}$$

where μ is an arbitrary integration constant. Hence, we have the complete freedom of choosing the value of μ in (6) as regards solving the field Eq. (5b), which leaves us to consider the vacuum energy. Writing $\Lambda_2(q)$ from Eq. (4b) as $\epsilon_2(q) - \mu\,q$, we then see that the huge vacuum energy density stored in $\epsilon_2(q)$ can be compensated by the counterterm $-\mu q$. This compensation occurs due to thermodynamic identities applied to the equilibrium self-sustained vacuum (we shall see this for the example of a freely-suspended film in Sec. 3). Here, μ plays the role of a chemical potential [5], which is self-tuned in the equilibrium state of the quantum vacuum. The self-tuning gives rise to the nullification of the vacuum energy density for the equilibrium vacuum in the absence of matter fields,

$$\Lambda_2(q_{\rm eq}) = \epsilon_2(q_{\rm eq}) - \mu_{\rm eq}\,q_{\rm eq} = 0. \tag{7}$$

No such compensation is expected for the vacuum of the fundamental scalar field ϕ. However, if both ϕ and q fields are present, then the energy $\epsilon_1(\phi)$ will also be compensated in the equilibrium vacuum by the readjustment of the chemical potential μ [4].

3. Freely suspended film. The thin film considered here is a two-dimensional (2D) object embedded into the three-dimensional (3D) Euclidean space $\mathbb{R}^3$ with Cartesian coordinates X, Y, Z. In modern parlance, this film is a 2D brane [19]. We are interested in a freely-suspended film, which is not surrounded by dense matter. In equilibrium, the film is parallel to the (X, Y)-plane, and this state is analogous to the 4D equilibrium Minkowski vacuum to be discussed in Sec. 4.

The Hamiltonian [10] which describes the deformations of the film – variation of the density of the film and bending (displacement) of the film – can be written in the following form:

$$H = \int d^2x\,\left[\sqrt{g}\,\epsilon\left(\frac{n}{\sqrt{g}}\right) - \mu\,n\right] + H_{\rm bending}. \tag{8}$$

Here, $n(x,y) \equiv \int dz\,n(x,y,z)$ is the 2D particle density obtained by integrating over the extra dimension of the bulk space $\mathbb{R}^3$. For simplicity, we assume the absence of folding. The total number of particles in the film, $N = \int d^2x\,n(x,y)$, is a conserved quantity. We introduce the corresponding Lagrange multiplier μ, which plays the role of a chemical potential. Particle conservation (or mass conservation in Ref. [10]) is the main condition for the existence of a stable freely-suspended film. Hence, the film belongs to the class of self-sustained systems.

The potential term (the first term in the integrand of Eq. (8)) depends on the quantity $n/\sqrt{g}$, the particle density per unit area of the curved film (recall $dS = \sqrt{g}\,dxdy$). Here and in Eq. (8), we have used the definition $g \equiv \det(g_{ik})$ for the curved-film metric g_{ik} with signature $(+,+)$. The second term on the right-hand side of Eq. (8) is the bending energy. It contains gradients of the metric, but, due to the lack of invariance under general coordinate transformations, it is not equivalent to the standard curvature term in gravitation theories. Still, it includes the Gauss curvature R, which is a total derivative in two dimensions. Here, we are not really interested in the second term $H_{\rm bending}$ of Eq. (8), since the central point of our argument will concern the first term, which will be seen to be equivalent to the vacuum energy. So, let us consider only the first term on the right-hand side of Eq. (8).

Variation of the Hamiltonian (8) over n gives the following equation:

$$\frac{d\epsilon}{dq} = \mu, \tag{9a}$$

$$q \equiv \frac{n}{\sqrt{g}}, \tag{9b}$$

and variation over $\sqrt{g}$ gives the surface tension σ of the film,

$$\sigma \equiv \frac{\delta H}{\delta\sqrt{g}} = \epsilon(q) - q\,\frac{d\epsilon}{dq} = \epsilon(q) - \mu\,q. \tag{10}$$

For the freely-suspended film (i.e., no forces from the environment), the surface tension is zero in equilibrium, $\sigma_{\rm eq} = 0$. The chemical potential μ is self-tuned to reach the equilibrium state. Since the variation over the metric g^{ik} gives the stress tensor $T_{ik} = -\sigma g_{ik}$ [10, 20, 21], the surface tension σ plays the role of a cosmological constant in the 2+0 gravity theory of the film, which is nullified in equilibrium.

As emphasized in Ref. [10], the equilibrium condition $\sigma = 0$ is not disturbed by quantum fluctuations, because it is the consequence of the thermodynamic identity $\sqrt{g}\,\epsilon - \mu\, n = -P$, where P is the external pressure (thermal effects are not considered). In the absence of external forces, the surface tension is zero irrespective of the quantum fluctuations. Of course, the quantum fluctuations contribute to the energy ϵ, and this contribution can be essential. But this contribution is always fully compensated by the counterterm $-\mu\, n$ in equilibrium, which is, in fact, the property of any self-sustained vacuum. This suggests that, if the vacuum of our Universe belongs to the class of self-sustained systems, its energy in equilibrium is fully cancelled in spite of the huge effects of vacuum fluctuations. The cancellation results in a zero value for the cosmological constant in an equilibrium Universe without ponderable matter.

4. 4D brane. The corresponding modification of the Einstein action on the four-dimensional (4D) "brane" is

$$I = -\int d^4x \sqrt{-g}\left[\epsilon\left(\frac{n}{\sqrt{-g}}\right) + \frac{R}{16\pi G_N} + \mathcal{L}^M[\psi]\right] + \\ + \mu \int d^4x\, n, \tag{11}$$

where the metric $g_{\mu\nu}$ has a Lorentzian signature $(-, +, +, +)$ making for a negative determinant $g \equiv \equiv \det(g_{\mu\nu})$ and where n is the 4D analog of the particle density of the 2D film (the 4D density n perhaps refers to the "atoms" of spacetime). In principle, the gravitational coupling G may also depend on n, $G = G(n)$, but we fix $G = G_N$, for simplicity. Similarly, we omit any n-dependence of the parameters in the matter Lagrange density $\mathcal{L}^M(\psi)$, where ψ stands for a generic matter field.

Variation of the action (11) over the density n gives the analog of Eq. (9),

$$\frac{d\epsilon}{dq} = \mu\,, \tag{12a}$$

$$q \equiv \frac{n}{\sqrt{-g}}. \tag{12b}$$

Variation of the action (11) over the inverse metric $g^{\mu\nu}$ gives the Einstein equation,

$$\frac{1}{8\pi G_N}\left(R_{\mu\nu} - \frac{1}{2}\, R\, g_{\mu\nu}\right) = \Lambda(q)\, g_{\mu\nu} + T^M_{\mu\nu}\,, \tag{13}$$

with the following vacuum energy density:

$$\Lambda(q) = \epsilon(q) - q\,\frac{d\epsilon(q)}{dq} = \epsilon(q) - \mu\, q, \tag{14}$$

where the last equality relies on result (12a).

The same equations can be obtained if the action (11) is expressed directly in terms of q, and if one varies over q and $g^{\mu\nu}$, assuming that q is a conserved quantity. This demonstrates the universal and generic properties of q-theory [4], which do not depend on the particular realization of the conserved quantity q.

5. Discussion. In this Letter, we have introduced an alternative description of q-theory, which is based on an analogy with the two-dimensional (2D) condensed-matter film. The freely-suspended film belongs to the class of self-sustained systems, which is characterized by a conservation law. For the case of a nonrelativistic 2D film, the relevant conservation law is the conservation of mass of the film. Due to this conservation law, the surface tension of the film is nullified in equilibrium, in spite of the large contributions of the vacuum fluctuations to the energy of the film. For the (3+1)-dimensional relativistic analog of such a film, the same type of conservation law leads to the nullification of the cosmological constant – the 3+1 analog of the 2D surface tension.

The 4D brane provides a new realization of q-theory. Recall that the main input of q-theory is the assumption that the quantum vacuum belongs to the class of self-sustained systems, characterized by the conservation law for some kind of "particle number." For a vacuum within this class, the nullification of the cosmological constant automatically takes place in equilibrium, while, out of equilibrium, the cosmological constant can be large and comparable with the estimates based on the summation of the zero-point energies of the quantum fields (with appropriate cutoffs).

Since the quantity $\sqrt{-g}$ enters the action as a separate variable, this 4D-brane realization of q-theory represents a generalization of unimodular gravity [1, 22–25]. Gravity in four dimensions is, of course, very different from gravity in two dimensions, for example, there are gravitational waves (gravitons) in four dimensions but not in two. For the 4D theory (11), we may then consider gravitational processes at energies far below the binding energy of the "atoms" of spacetime responsible for the number density n. For such low-energy processes,

n is effectively fixed and nondynamical, $n = n_0$. From the 4D general covariance of (11), we have that n is a scalar density with the same weight as the square root of the negative of the determinant of the metric. Introducing a prior metric with determinant g_0, we then have $n = n_0 \propto \sqrt{-g_0}$ and the q variable is effectively equal to $q \propto \sqrt{-g_0}/\sqrt{-g}$. The theory (11) written in terms of the inverse variable $\widetilde{\sigma} = \sqrt{-g}/\sqrt{-g_0}$ is essentially the one studied in Ref. [26], where the role of vacuum-matter energy exchange has been investigated.

Now, return to the original form of q as given by Eq. (12b). It can then be shown that, for homogeneous matter fields in a cosmological context, the vanishing covariant divergence of the vacuum-energy term on the right-hand side of Eq. (13) gives $\partial_t \Lambda = 0$, which implies $dq/dt = 0$. From the definition of q, we can then relate the rate of change of the metric determinant to the rate of change of the brane number density, $d(\ln\sqrt{-g})/dt = d(\ln n)/dt$. The role of this type of intra-brane dynamics needs to be clarified.

Let us also comment on the main difference between our approach and the one of Ref. [27], where the cosmological constant was estimated as $\sqrt{N}$, with N the number of elements. Reference [27] noted that the individual contributions to the action I have random signs and assumed that their sum vanishes on average, with residual fluctuations of order $\sqrt{N}$ being responsible for the observed value of Λ. In our case, the contribution to the action I is proportional to N, while the cancellation takes place for the quantity Λ which enters the Einstein equations. Λ does not necessarily coincide with the vacuum energy density ϵ, which enters the action. The difference between Λ and ϵ reflects two different definitions of the energy of quantum fields on an external time-independent background, see Ref. [28]. The first one defines the energy in terms of the stress-energy tensor, while the second one identifies the energy with the Hamiltonian.

According to the q-theory approach to the cosmological constant problem, the present small value of Λ is the result of the incomplete cancellation of the vacuum energy in a slowly evolving nonequilibrium Universe [5].

GEV thanks Efim Kats and Vladimir Lebedev for valuable discussions.

Note Added in Proof. An early paper on the vanishing surface tension of fluid membranes as an analog of the near-zero cosmological constant of general relativity is Ref. [29].

1. S. Weinberg, Rev. Mod. Phys. **61**, 1 (1989).
2. S. Weinberg, *Theories of the cosmological constant*, in: N. Turok, *Critical Dialogues in Cosmology*, World Scientific, Singapore (1997), p. 195; arXiv:astro-ph/9610044.
3. S. Nobbenhuis, Found. Phys. **36**, 613 (2006); arXiv: gr-qc/0411093.
4. F. R. Klinkhamer and G. E. Volovik, Phys. Rev. D **77**, 085015 (2008); arXiv:0711.3170.
5. F. R. Klinkhamer and G. E. Volovik, Phys. Rev. D **78**, 063528 (2008); arXiv:0806.2805.
6. F. R. Klinkhamer and G. E. Volovik, Phys. Rev. D **80**, 083001 (2009); arXiv:0905.1919.
7. F. R. Klinkhamer and G. E. Volovik, Phys. Rev. D **79**, 063527 (2009); arXiv:0811.4347.
8. F. R. Klinkhamer and G. E. Volovik, JETP Lett. **91**, 259 (2010); arXiv:0907.4887.
9. G. E. Volovik, *The Universe in a Helium Droplet*, Paperback Edition, Oxford UP (2008).
10. E. I. Kats and V. V. Lebedev, Phys. Rev. E **91**, 032415 (2015); arXiv:1501.06703.
11. M. J. Duff and P. van Nieuwenhuizen, Phys. Lett. B **94**, 179 (1980).
12. A. Aurilia, H. Nicolai, and P. K. Townsend, Nucl. Phys. B **176**, 509 (1980).
13. S. W. Hawking, Phys. Lett. B **134**, 403 (1984).
14. M. J. Duff, Phys. Lett. B **226**, 36 (1989).
15. M. J. Duncan and L. G. Jensen, Nucl. Phys. B **336**, 100 (1990).
16. R. Bousso and J. Polchinski, JHEP **0006**, 006 (2000); arXiv:hep-th/0004134.
17. A. Aurilia and E. Spallucci, Phys. Rev. D **69**, 105004 (2004); arXiv:hep-th/0402096.
18. Z. C. Wu, Phys. Lett. B **659**, 891 (2008); arXiv:0709.3314.
19. V. A. Rubakov, Phys. Usp. **44**, 871 (2001); arXiv: hep-ph/0104152.
20. V. V. Lebedev and A. R. Muratov, Sov. Phys. JETP **68**, 1011 (1989).
21. E. I. Kats and V. V. Lebedev, *Fluctuational Effects in the Dynamics of Liquid Crystals*, Springer, Berlin (1993).
22. J. J. van der Bij, H. van Dam, and Y. J. Ng, Physica A **116**, 307 (1982).
23. A. Zee, *Remarks on the cosmological constant problem*, in: *High-Energy Physics: Proceedings of Orbis Scientiae 1983*, ed by S. L. Mintz and A. Perlmutter, Plenum Press, N.Y. (1985), p. 211.
24. W. Buchmüller and N. Dragon, Phys. Lett. B **207**, 292 (1988).
25. M. Henneaux and C. Teitelboim, Phys. Lett. B **222**, 195 (1989).
26. F. R. Klinkhamer, arXiv:1604.03065.
27. M. Ahmed and R. D. Sorkin, Phys. Rev. D **87**, 063515 (2013); arXiv:1210.2589.
28. D. V. Fursaev, Phys. Rev. D **59**, 064020 (1999); arXiv:hep-th/9809049.
29. J. Samuel and S. Sinha, Phys. Rev. Lett. **97**, 161302 (2006); arXiv:cond-mat/0603804.