Fermi–Pasta–Ulam recurrence and modulation instability

E. A. Kuznetsov[1)]

Lebedev Physical Institute RAS, 119991 Moscow, Russia

Landau Institute for Theoretical Physics RAS, 119334 Moscow, Russia

Novosibirsk State University, 630090 Novosibirsk, Russia

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1. The phenomenon of recurrence in nonlinear systems with many degrees of freedom was first observed in the classical numerical experiments by Fermi, Pasta, and Ulam [1] in 1954. The idea of Fermi was to address how randomization due to the nonlinear interaction leads to the energy equipartition between large number of degrees of freedom in the mechanical chain. The chain in [1] had a quadratic nonlinearity and included 64 oscillators supplemented with long-wave initial conditions. Instead of the anticipated energy equipartition, numerical experiments showed that after a finite time a recurrence to the initial data was achieved accompanied by a quasi-periodic energy exchange between several initially exited modes. That recurrence phenomenon became known as the famous Fermi–Pasta–Ulam (FPU) problem and has been one of the central problems of the nonlinear science, studied in numerous papers. Many other peculiarities of this problem have been discovered and studied (for more details, see the original papers [2] published before the era of integrability for nonlinear systems).

Since the discovery of the Inverse Scattering Transform, which was first applied to the KDV equation [3], and later to the nonlinear Schrodinger equation (NLSE) [4], many aspects of the FPU recurrence became more clear. The main conjecture of many reviews devoted to this phenomenon (see, e.g. [5]) is that the recurrence is a pure nonlinear phenomenon mainly intrinsic to the integrable models. The FPU recurrence was also intensively studied experimentally (the first experimental demonstration in optical fibers of the FPU recurrence was presented in [6]).

The main goal of this paper is to explain how the recurrence phenomenon appears within the one-dimensional NLSE

$$i\psi_t + \psi_{xx} + 2|\psi|^2\psi = 0. \tag{1}$$

This equation with a reasonable accuracy describes propagation of optical solitons in fibers.

At the present time, there are known many exact solutions of the NLSE which describe propagation of solitons/breathers on the condensate background. Such solution was constructed for a first time in [7] and later in many other papers (see [8] and references therein). All these solutions show that after a while the condensate recovers its amplitude but has a different (but constant) phase. This is the analogue of the FPU recurrence for the NLSE (see recent numerical confirmations [9] for arbitrary initial conditions). In this paper we give a qualitative explanation of the FPU analog for cnoidal waves. For fiber communications the latter means that such FPU recurrence can ensure the preservation of information, in spite of the MI existence.

2. To find the cnoidal wave to Eq. (1) one needs to seek for a stationary solution in the form $\psi = e^{i\lambda t}\psi_0(x)$ assuming $\psi_0(x)$ to be real. By introducing intensity $I = \psi_0^2$ and shifting, $I = -[\wp(x - i\omega') - \lambda^2/3]$, the stationary NLSE transforms into the equation for the elliptic Weierstrass function: $(\wp')^2 + U_\wp = 0$, where $U_\wp = -4(\wp - e_1)(\wp - e_2)(\wp - e_3)$ has a meaning of a "potential energy" for trajectories related to an "energy" equal to zero. Here $e_{1,2,3}$ are values of $\wp$ in points $z = \omega$, $\omega + i\omega', i\omega'$. The Weierstrass elliptic function is known as a double-periodic analytical function with periods 2ω (along real axis) and $2i\omega'$ (along imaginary axis). Oscillations between zero points e_2 and e_3 in the "potential" $U_\wp$ defines the real period 2ω. The oscillations between e_1 and e_2 in imaginary "time" $(x \to iy)$ yields another period $2i\omega'$. As known [10] (see also [11]), the Weierstrass elliptic function can be represented in the form of the solitonic lattice. In particular, intensity I reads as follows

$$I = \mu^2 \sum_{n=-\infty}^{\infty} \{\operatorname{sech}^2[\mu(x - 2n\omega)] + \operatorname{cosech}^2[\mu(2n-1)\omega]\},$$

where $\mu = \pi/(2\omega')$. Hence in the limit of large spatial period we arrive at the soliton $\psi_0 = \lambda \operatorname{sech}[\lambda(x - x^{(0)})]$,

1)e-mail: kuznetso@itp.ac.ru

and in the opposite limit, $\omega'/\omega \to \infty$, due to the soliton *overlapping* the cnoidal wave transforms into the condensate solution $\psi_0 = \lambda/\sqrt{2}$.

3. It is well-known that the condensate solution is unstable relative to the modulation instability with growth rate $\gamma = k\sqrt{2\lambda^2 - k^2}$. It is less known that the cnoidal wave is also unstable relative to the MI [12]. To find the growth rate in this case one needs to solve the NLSE linearized on the background of the cnoidal wave that results in the system of coupled PDEs of the *second* order. As it was shown in [12], using the linearized version of the Zakharov–Shabat dressing procedure [13] simplifies significantly solution of this linear problem, which reduces to solution of the *first* order PDEs. This version, in fact, was introduced for a first time in 1974 [11] and developed later in [14]. In the case of the cnoidal wave this procedure yields the answer written in the implicit form [12]. For the condensate it gives the growth rate presented above. When the distance between solitons becomes large enough and, respectively, overlapping between solitons weak, the maximal instability growth rate occurs to be exponentially small, but increases with the distance decrease [12]: $\gamma_{\max} = 8\,(\pi/\omega')^2 \exp\left(-\pi\omega/\omega'\right)$.

4. What happens at the nonlinear stage of the MI for the cnoidal wave? We know that, according to Zakharov and Shabat [4], the phase space of the NLSE represents discrete number of degrees of freedom which are solitons (these are the most nonlinear objects) and solutions corresponding to the continuous spectrum. Moreover, we know that collisions between solitons are elastic and pairwise. A scattering of two solitons results only in changing of two their parameters. For the first soliton the center of soliton mass $x_1^{(0)}$ and its phase $\phi_1^{(0)}$ get the following shifts:

$$\Delta x_1^{(0)} = \frac{1}{2\eta_1}\log\left|\frac{\lambda_1 - \lambda_2^*}{\lambda_1 - \lambda_2}\right|^2, \ \Delta\phi_1^{(0)} = 2\arg\left(\frac{\lambda_1 - \lambda_2^*}{\lambda_1 - \lambda_2}\right),$$

where $\lambda_{1,2}$ are eigenvalues corresponding to the first and second solitons, $\eta = \operatorname{Im}\lambda > 0$ and λ^* means complex conjugation of λ. Analogous formula can be written for the second soliton.

Because the cnoidal wave has the form of the solitonic lattice, any soliton from the lattice after interaction with a soliton propagating along the cnoidal wave will undergo the same shift for its center of mass and phase. This means that after scattering of the propagating soliton with the lattice, the cnoidal wave will restore its previous form (up to the definite spatial and phase shifts). Evidently, the same statement will be valid for condensate as the partial solution of the cnoidal wave. The interaction of condensate with any soliton after its propagation will restore amplitude of the condensate but its (constant) phase will be different from the initial value.

Scattering of a soliton with the non-soliton part also retains the soliton form unchanged except shifts of both center of mass of the soliton and its phase. Thus, the cnoidal wave subject to the modulation instability, at the nonlinear stage of the modulation instability development, should recover its form together with some phase and spatial shifts remaining its original coherence. This is the qualitative explanation of the FSU recurrence for the cnoidal wave and for the condensate, in particular. It is necessary to underline that the same phenomenon takes place for the KDV cnoidal wave that was found in 1974 [11].

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