Pis'ma v ZhETF, vol. 105, iss. 7, pp. 442 – 443

Type-III and IV interacting Weyl points

J. Nissinen[+1)], G. E. Volovik[+*1)]

[+]Low Temperature Laboratory, Aalto University, P.O. Box 15100, FI-00076 Aalto, Finland

[*]Landau Institute for Theoretical Physics, 142432 Chernogolovka, Russia

Submitted 7 March 2017

DOI: 10.7868/S0370274X17070062

Weyl fermions [1] are massless fermions whose masslessness (gaplessness) is topologically protected [2–4]. They appear in the Standard Model of fundamental interactions and as quasi-relativistic quasiparticles in topological semimetals [5–11] and chiral superfluid ^{3}He-A [12, 13]. A Weyl point is a topologically protected band touching in momentum space described by a 2×2 effective Weyl Hamiltonian with linear dispersion. The most general Hamiltonian describing Weyl fermions at a Weyl point at $\mathbf{p} = 0$ is

$$H(\mathbf{p}) = e^i_0 \sigma^0 p_i + e^i_a \sigma^a p_i, \tag{1}$$

where p_i are the components of three-momentum $\mathbf{p}$, $\sigma^0 = \mathbf{1}$ and σ^a for $a = 1, 2, 3$ are the Pauli matrices. The 12 coefficients e^i_0 and e^i_a in the linear expansion are equivalent to the corresponding components of the tetrad field e^μ_α in general relativity (GR), where $\mu, \alpha = 0, 1, 2, 3$ in 3+1 dimensions. Depending on these parameters, the Weyl fermions associated with the Weyl point form two distinct classes [14, 15], called type-I and type-II [16]. In the latter case, the Weyl point is a topologically protected touching of the electron and hole pockets in a semimetal. In GR such a fermionic spectrum can emerge behind a black hole horizon [17, 18].

Here we suggest that for the interacting fermions the situation can be more complicated: The Green's function contains all 16 components e^μ_α of the tetrad field in an expansion around the Weyl point in 4-momentum space $p_\mu = (\omega, \mathbf{p})$. This leads to two additional distinct classes of Weyl points, type-III and type-IV, depending on the four signs of the elements g^{00} and g_{00} of the effective tetrad metric. This allows us to simulate additional solutions of Einstein equations at interfaces where the components of the metric change sign, including spacetimes with closed timelike curves.

Interacting fermions. A given metric is expressed in terms of the tetrad field as

$$g^{\mu\nu} = \eta^{\alpha\beta} e^\mu_\alpha e^\nu_\beta, \tag{2}$$

where the (dimensionless) metric $\eta^{\alpha\beta} = \mathrm{diag}(1, -1, -1, -1)$ has Lorentzian signature. In the 2×2 Weyl Hamiltonian Eq. (1), as well as in the 4×4 Dirac Hamiltonian, the elements e^0_a and e^0_0 of tetrad fields are missing, since p_0 of the four vector $p_\mu = (\omega, \mathbf{p})$ is absent. The frequency $p_0 \equiv \omega$ appears in the Green's function $G(\omega, \mathbf{p})$. For noninteracting fermions, the Green's function

$$G^{-1}(\omega, \mathbf{p}) = H - \omega = e^0_0 \sigma^0 p_0 + e^i_0 \sigma^0 p_i + e^i_a \sigma^a p_i \tag{3}$$

contains the term e^0_0 with fixed $e^0_0 = -1$, but the term with e^0_a is still missing. This term may appear in an interacting system, e.g. from the fermion self-energy, where the general form of the 2×2 Green's function is:

$$G^{-1}(p_\mu) = e^0_0 \sigma^0 p_0 + e^i_0 \sigma^0 p_i + e^i_a \sigma^a p_i + e^0_a \sigma^a p_0. \tag{4}$$

The effective metric Eq. (2) for the Weyl (and massless Dirac) fermions is determined by the poles of the Green's function via the dispersion relation

$$g^{\mu\nu} p_\mu p_\nu = 0. \tag{5}$$

The emergence of the tetrad elements e^0_a in an interacting system may drastically change the behavior of the fermions. First, the matrix element $g^{00} = (e^0_0)^2 - \mathbf{e}^0 \cdot \mathbf{e}^0$ in the fermion dispersion relation may now cross zero and become negative. Second, the matrix element $g^{0i} = e^0_0 e^i_0 - \mathbf{e}^0 \cdot \mathbf{e}^i$ contains the two tetrad vectors e^0_a and e^i_0. For clarity, we consider the case where e^i_a is isotropic and characterized by the "speed of light" or Fermi-velocity c of an isotropic Weyl cone, with e^i_0, e^0_a parametrized in terms of the two velocity fields $\mathbf{v}$ and $\mathbf{w}$, respectively:

$$e^0_0 = -1, \;\; e^i_a = c\delta^i_a, \;\; e^i_0 = v^i, \;\; e^0_a = w_a/c. \tag{6}$$

The corresponding effective metric $g^{\mu\nu}(\mathbf{v}, \mathbf{w})$ is

$$g^{00} = 1 - \frac{w^2}{c^2}, \; g^{0i} = -(w^i + v^i), \; g^{ij} = -c^2\delta^{ij} + v^i v^j. \tag{7}$$

1) e-mail: jaakko.nissinen(at)aalto.fi; volovik(at)ltl.ttk.fi

The line-interval $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$ is given by the inverse $g_{\mu\nu}$ with determinant $g = -1/e^2$, where

$$e = \det e^\mu_\alpha = -c(c^2 + \mathbf{v}\cdot\mathbf{w}). \quad (8)$$

The dispersions relations are

$$\omega_\pm(\mathbf{p}) = \frac{-g^{0i}p_i \pm \sqrt{g^{00}(-g^{ij}p_ip_j) + (g^{0i}p_i)^2}}{g^{00}}. \quad (9)$$

The time-like components $g_{00} = c^4(c^2 - v^2)/g$ and $g^{00} = 1 - w^2/c^2$ indicate singular behavior in the Weyl spectrum upon change of sign in analogy with solutions of GR.

From type-I to type-II. For $w^2 < c^2$ and $v^2 < c^2$ one has a type-I Weyl point with tilted cone. For $v^2 > c^2$, the components g_{00} and g^{vv} of the metric change sign, forming the type-II Weyl point. The interface $v^2 = c^2$ between the type-I and type-II Weyl semimetals serves as the analog of either the ergosurface or the horizon of a blackhole [18], depending on the orientations of the velocity vectors $\mathbf{v}, \mathbf{w}$ and the interface $v^2 = c^2$.

From type-I to type-III. Let us fix $\mathbf{v} = 0$:

$$\omega_\pm(\mathbf{p}) = \frac{1}{1 - \frac{w^2}{c^2}}\left(wp_\parallel \pm \sqrt{c^2p_\parallel^2 + (c^2 - w^2)p_\perp^2}\right), \quad (10)$$

where $p_\parallel$ and $p_\perp$ are defined with respect to $\mathbf{w}$. For $w < c$ one has the type-I Weyl point with tilted cone. For $w > c$, both g^{00} and $\omega_\pm$ change sign and the frequency has both positive and negative imaginary parts implying an instability or localization in the directions transverse to $\mathbf{w}$ (see Ref. [13] for a bosonic example). The momentum space surface separating the propagating fermionic states from states with complex spectrum forms the cone $p_\parallel = \pm p_\perp\sqrt{\frac{w^2}{c^2} - 1}$. We call spectra with $w^2 > c^2$ and $v^2 < c^2$, i.e. $g^{00} < 0$ and $g_{00} > 0$, type-III Weyl. With Weyl points of type III, one may simulate the closed timelike curves in GR [19], i.e. curves $f(\tau)$ satisfying $\dot{f}_\mu \dot{f}_\nu g^{\mu\nu} > 0$. An axial velocity field $\mathbf{w} = w(r)\hat{\boldsymbol{\phi}}$ and $\mathbf{v} = 0$ produces the line-interval

$$ds^2 = \left(dt - \frac{1}{c^2}\mathbf{w}\cdot d\mathbf{x}\right)^2 - \frac{1}{c^2}d\mathbf{x}^2. \quad (11)$$

At $w^2(r) > c^2$, one has $g_{\phi\phi} = \frac{r^2}{c^2}(\frac{w^2(r)}{c^2} - 1) > 0$ and closed timelike curves $t = \text{const}$, $z = \text{const}$, $r = \text{const}$.

Type-IV Weyl. A distinct type-IV spectrum in the region with both $g^{00} < 0$ and $g_{00} < 0$ also emerges. The transitions between type-I, type-II, type-III and type-IV Weyl points are peculiar forms of Lifshitz transitions.

Transition in the tetrad field. There is also a Lifshitz transition which is not manifested in the metric $g^{\mu\nu}$, but is seen in the tetrad field. This happens when the tetrad determinant in Eq. (8) crosses zero at $\mathbf{v}\cdot\mathbf{w} = -c^2$, and the right-handed fermion transforms to the left-handed one.

This work has been supported by the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (Grant Agreement # 694248).

Full text of the paper is published in JETP Letters journal. DOI: 10.1134/S0021364017070013

1. H. Weyl, Z. Phys. **56**, 330 (1929).
2. H. B. Nielsen and M. Ninomiya, Nucl. Phys. B **185**, 20 (1981); Nucl. Phys. B **193**, 173 (1981).
3. P. G. Grinevich and G. E. Volovik, J. Low Temp. Phys. **72**, 371 (1988).
4. P. Hořava, Phys. Rev. Lett. **95**, 016405 (2005).
5. C. Herring, Phys. Rev. **52**, 365 (1937).
6. A. A. Abrikosov and S. D. Beneslavskii, JETP **32**, 699 (1971).
7. X. Wan, A. M. Turner, A. Vishwanath, and S. Y. Savrasov, Phys. Rev. B **83**, 205101 (2011).
8. A. A. Burkov and L. Balents, Phys. Rev. Lett. **107**, 127205 (2011).
9. B. Q. Lv, H. M. Weng, B. B. Fu, X. P. Wang, H. Miao, J. Ma, P. Richard, X. C. Huang, L. X. Zhao, G. F. Chen, Z. Fang, X. Dai, T. Qian, and H. Ding, Phys. Rev. X **5**, 031013 (2015).
10. S.-Y. Xu, I. Belopolski, N. Alidoust et al. (Collaboration), Science **349**, 613 (2015).
11. G. Chang, S.-Y. Xu, D. S. Sanchez, S.-M. Huang, C.-C. Lee, T.-R. Chang, G. Bian, H. Zheng, I. Belopolski, N. Alidoust, H.-T. Jeng, A. Bansil, H. Lin, and M. Z. Hasan, Sci. Adv. 2016; 2 : e1600295 (2016).
12. T. D. C. Bevan, A. J. Manninen, J. B. Cook, J. R. Hook, H. E. Hall, T. Vachaspati, and G. E. Volovik, Nature **386**, 689 (1997).
13. G. E. Volovik, *The Universe in a Helium Droplet*, Clarendon Press, Oxford (2003)
14. G. E. Volovik and M. A. Zubkov, Nuclear Physics B **881**, 514 (2014).
15. Y. Xu, F. Zhang, and C. Zhang, Phys. Rev. Lett. **115**, 265304 (2015).
16. A. A. Soluyanov, D. Gresch, Z. Wang, Q. S. Wu, M. Troyer, X. Dai, and B. A. Bernevig, Nature **527**, 495 (2015).
17. P. Huhtala and G. E. Volovik, ZhETF **121**, 995 (2002); JETP **94**, 853 (2002).
18. G. E. Volovik, JETP Lett. **104**, 645 (2016).
19. E. Kajari, R. Walser, W. P. Schleich, and A. Delgado, Gen. Rel. Grav. **36**, 2289 (2004).