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Calogero-Sutherland system with two types interacting spins

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The Calogero-Sutherland (CS) model [1–4] describes one-dimensional system of interacting pairwise particles through long range potentials. It has a lot of applications, in particular with the quantum Hall effect [5], matrix models [6], and orthogonal polynomials [7]. In this paper we consider the classical case. The classical CS model is an integrable system in the Liouville sense. Moreover, it remains integrable if one adds the so-called spin variables. The resulting system has form of the Euler–Arnold $\mathrm{SL}(N)$ top with the inertia tensor depending on the positions of interacting particles [8, 9].

Denote the coordinates of the particles $\mathbf{u} = (u_1, \ldots, u_N)$, their momenta $\mathbf{v} = (v_1, \ldots, v_N)$ and the spin variables $\{S_{jk}\}$ arranged into matrix $S = \sum_{ij}^N E_{ij} S_{ij}$ (here $\{E_{ij}\}$ is the standard basis in $\mathrm{Mat}(N)$, i.e. $(E_{ij})_{ab} = \delta_{ia}\delta_{bj}$). The latter is an element of the Lie algebra $\mathrm{sl}(N)$. The spin CS model is described by the Hamiltonian

$$H^{\mathrm{CS}} = \frac{1}{2}\sum_{j=1}^N v_j^2 - \sum_{j<k} \frac{S_{jk}S_{kj}}{\sinh^2(u_j - u_k)}. \qquad (1)$$

The Poisson brackets between positions of particles and momenta are canonical $\{v_k, u_j\} = \delta_{jk}$, while the Poisson structure for $\{S_{jk}, S_{mn}\}$ is given by the Dirac brackets. They can be obtained starting from the Lie–Poisson brackets on the Lie coalgebra $\mathrm{sl}^*(N)$ after imposing constraints $S_{\mathrm{diag}} = 0$ (and some gauge fixation) resulting from the coadjoint action of the diagonal subgroup of $\mathrm{SL}(N)$ on the spin variables S. The case when $S \in \mathrm{SO}(N)$ is known as well [10]. Some further generalizations can be found in [11].

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Our generalization of (1) is as follows:

$$H = \frac{1}{2}\sum_{j=1}^N v_j^2 + \sum_{j<k} \frac{S_{jk}^2 + T_{jk}^2 - 2S_{jk}T_{jk}\cosh(u_j - u_k)}{\sinh^2(u_j - u_k)}, \qquad (2)$$

where S_{ij}, T_{ij} are elements of antisymmetric matrices S and T ($S_{jk} = -S_{kj}$, $T_{jk} = -T_{kj}$) with the Lie–Poisson brackets on the direct sum of two Lie coalgebras $\mathrm{so}^*(N) \oplus \mathrm{so}^*(N)$:

$$\begin{gathered} \{S_{ij}, S_{kl}\} = -\frac{1}{2}(S_{il}\delta_{kj} - S_{kj}\delta_{il} - S_{ik}\delta_{lj} + S_{lj}\delta_{ik}), \\ \{T_{ij}, T_{kl}\} = \frac{1}{2}(T_{il}\delta_{kj} - T_{kj}\delta_{il} - T_{ik}\delta_{lj} + T_{lj}\delta_{ik}), \\ \{S_{ij}, T_{kl}\} = 0. \end{gathered} \qquad (3)$$

The phase space $\mathbb{R}^{2N-2} \times \mathcal{O}_{\mathrm{SO}(N)} \times \mathcal{O}_{\mathrm{SO}(N)}$ consists of $\mathbb{R}^{2N-2}$ parameterized by momenta and positions of N particles in the center of mass frame and two coadjoint orbits $\mathcal{O}_{\mathrm{SO}(N)}$. Each orbit is obtained from $\mathrm{so}^*(N)$ by fixation of the Casimir functions (i.e. the eigenvalues of matrices S and T). The Poisson structure (3) keeps the same form on $\mathcal{O}_{\mathrm{SO}(N)} \times \mathcal{O}_{\mathrm{SO}(N)}$. The dimension of generic $\mathrm{SO}(N)$ orbit Equals[2)] $(1/2)(N^2 - N) - [N/2]$. Therefore, the dimension of the total phase space is $(N-1)(N+2) - 2[N/2]$.

For $N = 2$ the Lie algebra $\mathrm{so}(2)$ is commutative and the spin variables are fixed. In this case we obtain from (2) the Hamiltonian with two constants

$$H = \frac{v^2}{2} + \frac{m_1^2 + m_2^2 - 2m_1 m_2 \cosh(2u)}{\sinh^2(2u)}, \qquad (4)$$

which reproduces the CS model of the BC_1 type [3, 4].

[2)] $[x]$ stands for integer part of x.

We prove that there exists $(1/2)(N-1)(N+2)-[N/2]$ independent integrals of motion in involution, and the Hamiltonian (2) is one of them. To this end we construct the Lax pair and the classical r-matrix. Similarly to $\mathrm{SO}(N)$ version of (1) [10] (and in contrast to the $\mathrm{SL}(N)$ spin CS system) the M-operator can be explicitly constructed because the spin variables S, T are skew-symmetric matrices, and the additional reduction is not needed.

We derive the generalized CS system (GCS) using the Hitchin approach [12, 13]. The Lax operator of integrable system satisfies the Hitchin equations. They come from the self-duality equations in four dimensions after their reduction to two dimensional Riemann surface. Namely, instead of $\mathbb{R}^4$ one consider the four-dimensional space $\mathbb{R}^2\times\Sigma$, where Σ plays the role of the base spectral curve. The field content of the Hitchin system comes from the four-dimensional vector-potentials independent on the first two coordinates. One of them is the Higgs field that plays the role of the Lax operator of the integrable system. The coordinates of particles describe the moduli of solutions of the Hitchin equations, while the spin variables are the residues of the Higgs fields at the singular points. On Fig. 1 and Fig. 2 (see the full text) the base spectral curves Σ of CS and GCS systems are depicted. The Hitchin systems on singular curve (and, in particular, CS system) were studied previously in [14, 15].

Another important ingredient of the our construction is the so-called quasi-compact structure of the gauge group. It means that the gauge transformations at the singular points on the base spectral curve are reduced to the unitary group[3)]. We will come to this structure in relation to integrable systems elsewhere. As a result the spin variables become elements of the unitary algebra $\mathrm{SU}(N)$. To come to the integrable case we further reduce them to the orthogonal algebra $\mathrm{SO}(N)$.

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[3)] In the standard approach to the Hitchin systems the gauge group may have the quasi-parabolic structure, i.e. the gauge group is reduced at singular points to the triangular subgroup.