Pis'ma v ZhETF, vol. 106, iss. 10, pp. 607 – 608

On $6j$-symbols for symmetric representations of $U_q(\mathfrak{su}_N)$

A. Mironov[a,b,c,d1)], A. Morozov[b,c,d1)], A. Sleptsov[b,c,d,e1)]

[a]Lebedev Physics Institute, 119991 Moscow, Russia

[b]Alikhanov Institute of Theoretical and Experimental Physics (ITEP), 117218 Moscow, Russia

[c]Institute for Information Transmission Problems, 127994 Moscow, Russia

[d]National Research Nuclear University MEPhI, 115409 Moscow, Russia

[e]Laboratory of Quantum Topology, Chelyabinsk State University, 454001 Chelyabinsk, Russia

Submitted 18 October 2017
Resubmitted 23 October 2017

DOI: 10.7868/S0370274X17220027

Theory of the Racah–Wigner coefficients ($6j$-symbols) is among the standard topics in theoretical physics textbooks including the celebrated *Quantum Mechanics* of L. Landau and E. Lifshitz. It is of course a well known story in representation theory, because the $6j$-symbols intertwine the triple tensor products of representations $(R_1 \otimes R_2) \otimes R_3 \longrightarrow R_4$ and $R_1 \otimes (R_2 \otimes R_3) \longrightarrow R_4$. They are matrices

$$\left\{ \begin{array}{ccc} R_1 & R_2 & R_i \\ R_3 & R_4 & R_j \end{array} \right\},$$

with i and j labeling representations in the channels $R_1 \otimes R_2 = \oplus_i R_i$ and $R_2 \otimes R_3 = \oplus_j R_j$ respectively.

Tensor products are widely used in different topics of theoretical and mathematical physics from quantum mechanics to knot theory. Often needed are explicit formulas, because one typically wants to explicitly construct either the particle states or solutions to Yang–Baxter equations, i.e. the quantum $\mathcal{R}$-matrices. Therefore the Racah matrices were a subject of intensive investigation during the last three decades, still, surprisingly few results were obtained, until the very recent advances, which came from the newly discovered arborescent calculus and differential expansions of knot polynomials. These approaches allowed one to calculate many Racah matrices in various representations, but they are not yet brought into analytic form, i.e. all matrix elements are explicitly listed, but not described by a general formula with arbitrary i and j. In fact, getting such analytic formulas appears to be a separate non-trivial problem, and, in this letter, we address it in the very simple case of symmetric representations R_i, described by the single-line Young diagrams $[r_i]$ of length r_i, and their $\mathfrak{su}_N$-conjugates $\bar{R}_i$, described by the diagrams $[r_i^{N-1}]$ with $N-1$ lines of the same length. Somewhat surprisingly even in this case the answer was long known for $\mathfrak{su}_2$, but not for generic $\mathfrak{su}_N$. We perform this extension from 2 to N and use this example to describe the main ideas, which can hopefully lead to generalizations for non-symmetric representations (pure antisymmetric case is related to pure symmetric by the simple transformation $q \to -1/q$).

In arborescent calculus, we have only two Racah matrices $\bar{S}$ and S, which are fusion matrices of conformal blocks in WZW model. Quantum $6j$-symbols are proportional to elements of Racah matrices. Their explicit formulas for symmetric representations of $U_q(\mathfrak{su}_N)$ are known as some partial sums. We write them as balanced terminating q-hypergeometric series ${}_4\phi_3\left(\ldots|z=q^2\right)$. In order to obtain alternative expressions for the $6j$-symbols, one can use Sears' transformations. In particular, we find a representation in terms of orthogonal polynomials known as q-Racah (or Askey–Wilson for its continuous version) in the variable $q^{2(i-r)}+q^{2(1-r-i-N)}$ of degree $r-j$:

$$\left\{ \begin{array}{ccc} r & \bar{r} & i \\ r & \bar{r} & j \end{array} \right\} = \frac{[i]![j]![N-1]![N-2]!}{[i+N-2]![j+N-2]!} \times$$
$$\frac{[r]![r+N-2]![2r+N-1]!}{[r-i]![r-j]![r+i+N-1]![r+j+N-1]!} \times$$
$$\mathcal{R}_{r-j}(q^{2(i-r)}+q^{2(1-r-i-N)}|-r-1, 1-r-N, -2r-N, 0|q^2),$$
$$\left\{ \begin{array}{ccc} r & r & i \\ \bar{r} & \bar{r} & j \end{array} \right\} =$$
$$\frac{[r]!^2[N-1]![N-2]![2r+N-1]!}{[r-i]![r-j+N-2]![r+i+N-1]![r+j+N-1]!} \times$$
$$\mathcal{R}_{r-j}(q^{2(i-r)}+q^{2(1-r-i-N)}|-r-1, -r-1, -2r-N, 0|q^2).$$

1)e-mail: mironov@lpi.ru; mironov@itep.ru; morozov@itep.ru; sleptsov@itep.ru

Our main result is an extension of the Racah-matrix interpretation from the 1-parametric sub-family of such polynomials to two 2-parametric ones, by introducing the second parameter N (from $\mathfrak{su}_N$) in addition to r, which describes the symmetric representation.

For a system of q-hypergeometric polynomials to be orthogonal, they should satisfy a 3-term relation, what requires some art and imposes additional restrictions. In the case of ${}_4\phi_3(z)$, it is fixing z and the balanced series condition. In our cases these conditions are satisfied and we find the explicit form of 3-term relations with the help of our hypergeometric formulas. In the case of Racah polynomials, the 3-term relations possess an additional interpretation: they are nothing but the pentagon (Biedenharn–Elliot) identity, which reflects associativity of the Tanaka–Krein algebra of representations.

The main problem for Racah calculus is to go beyond the symmetric representation, in particular to find analytic expressions for already known Racah matrices S and $\bar{S}$ in various *two*-line representations, especially, in the rectangular ones, where there are no multiplicities and no associated ambiguities with the choice of bases in arborescent calculus. Direct attempts to guess such formulas as interpolating between the known matrix elements $\bar{S}_{ij}$ for particular i and j are somewhat tedious, especially because the number of summations is unknown. At the same time, in hypergeometric and Macdonald calculi (The Askey–Wilson or q-Racah polynomials can be described as simplest, one-variable, symmetric polynomials for the systems of roots of the BC_n type), there are natural ways for generalizations to higher representations, and this can significantly simplify the problem.

Also, the hypergeometric functions possess integral representations, which can be interpreted as correlators of conformal blocks within the Dotsenko–Fateev formalism, which is conceptually interesting. We remind that the Racah matrices naturally describe modular transformations of the conformal blocks, while the fact that matrices of the transformations of some objects can be also considered as the same objects themselves is intriguing and promising.

Full text of the paper is published in JETP Letters journal. DOI: 10.1134/S0021364017220040