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# Entanglement spectrum in superfluid phases of $^3$He

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The entanglement spectrum describes topological order in terms of the system's bulk properties without referring to the boundary. Counter-intuitively, for certain systems the entanglement spectrum in the ground state provides information about energy excitations. We analyze the entanglement spectrum in superfluid phases of $^3$He, the 3D B-phase and the planar phase in two dimensions. We find explicitly the wave functions of the low-lying eigenstates, including Majorana zero modes, as well as the corresponding part of the spectrum of the entanglement Hamiltonian.

Topological states of matter are often characterized via their boundary excitations. In particular, topological insulators and superconductors are frequently described as materials, which are gapped in the bulk, but have stable gapless excitations at the boundary. However, properties of such boundary states may depend on a particular boundary and its fine structure, whereas the essential properties of the material under study should be visible in its bulk characteristics.

The so called *bulk-boundary correspondence* establishes connections between the boundary states and the properties in the bulk [1]. To describe topological properties in the bulk in a more universal manner, without referring to the boundary structures, or to characterize systems without a boundary, it was suggested that one could use entanglement between its parts. The simplest characteristics of entanglement include the entanglement entropy, that is the entropy of the reduced density matrix of a subsystem. Of particular relevance are the scaling properties of the entanglement entropy with the size (area, length, depending on dimensions) of the boundary between the subsystems. In a gapped system one expects the so called area law, which predicts linear scaling and corresponds to the "short-range" part of the entanglement, localized near the border. Kitaev and Preskill [2] as well as Levin and Wen [3] suggested that the subleading term could be used to describe the topological order in the system.

It was later suggested [4] that one could use a more complete description, the full spectrum of the reduced density matrix of a subsystem, which contains more information. Li and Haldane noticed [4] in their analysis of a $\nu = 5/2$ FQHE system that this spectrum is reminiscent of the excitation spectrum at a real boundary. They put forward a conjecture that this similarity is a general property, which was later confirmed for a number of situations, however, in most cases analysis of the entanglement spectrum is numerical, cf. Refs. [5–9].

Note that the full entanglement spectrum contains more information than just the entanglement entropy. In certain cases it allows one to distinguish different topological phases, which have the same entanglement entropy [8]. It was also suggested that the entanglement spectrum may be more robust than the spectrum of real edge modes [6]. When the Haldane conjecture (or the *edge-entanglement correspondence*) holds, the entanglement spectrum in the ground state may reveal properties of excitations in the system. This is an unusual property, and it remains under debate how general it can be [8, 9].

Here we analyze the entanglement spectrum of the phases of superfluid $^3$He, which are topological superfluids: $^3$He-B in 3D and the planar phase in 2D. These phases are known to be topologically non-trivial insulators, both with topological charge $N = 1$ and, correspondingly, with one branch of gapless excitations at the boundary, cf. Ref. [10] and references therein. Here we demonstrate that Haldane's conjecture holds for these phases by explicit demonstration of gapless branches in the entanglement spectrum. Moreover, while Refs. [5, 6] found that the conjecture holds for quadratic Hamiltonians, we find explicitly the zero-pseudoenergy states and the spectrum at low pseudo-energies. These explicit results may be useful for further analysis.

We consider an infinite system, with a single-particle Hamiltonian $H$, which is divided into two parts by an imaginary plane $x = 0$. Having in mind the structure of the boundary states [11], we are looking for superpositions of states close to the Fermi surface; we find that

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this is indeed justified. Since the momentum $\mathbf{k}_\perp$, parallel to the plane, commutes with the mentioned operators, we may look for eigenstates with given $\mathbf{k}_\perp$. Then, for $|\mathbf{k}_\perp| < k_\mathrm{F}$, there are two regions of the values of $k_x$, close to Fermi surface: near $\pm k_\mathrm{F}^x \equiv \pm\sqrt{k_\mathrm{F}^2 - \mathbf{k}_\perp^2}$. For each spin component, we shall construct two eigenstates, one for each of these regions. We assume the simplest form of the superfluid $p$-wave order parameter in $^3$He-B, $A_{\alpha i} = \Delta_0 \delta_{\alpha i}$.

Based on the Haldane conjecture, we expect that the entanglement spectrum of $^3$He-B has a zero-(pseudo)energy state. To find it explicitly, we search for the corresponding eigenstates of the entanglement Hamiltonian using its representation as a Fourier transformation via the eigenstates of the Bogolyubov–de-Gennes Hamiltonian. This allows us to look for the wave functions using their analytical properties. We find four eigenstates, one for each spin component $\sigma = \pm$ and for the vicinity of each Fermi momentum, $k \approx s k_\mathrm{F}$ with $s = \pm$. These four zero modes of the entanglement Hamiltonian are given by

$$g_s^\sigma \propto e^{\mathrm{i} s k_\mathrm{F} x} \begin{pmatrix} 1 \\ -\mathrm{i} s\sigma \end{pmatrix} \frac{e^{-\Delta_0 x / v_\mathrm{F}}}{\sqrt{x}} \theta(x) \, |\sigma\rangle \,. \qquad (1)$$

Thus near each $\pm k_\mathrm{F}$ we find explicitly two degenerate Majorana zero modes of the entanglement Hamiltonian.

At small but finite $\mathbf{k}_\perp$ ($k_\perp \ll k_\mathrm{F}$) the zero level of the entanglement spectrum splits. This occurs via an emergent matrix element between two eigenstates, corresponding to two different spin components (on the same side of the 1D Fermi sea, say, near $+k_\mathrm{F}$).

We find that the low-lying part of the spectrum is described by an effective Hamiltonian for "relativistic" fermions,

$$2(1 - 2\hat{n}_\mathrm{F}(H_E)) = (\mathbf{k}_\perp \times \hat{\boldsymbol{\sigma}})_x \frac{\pi}{k_\mathrm{F} \ln(E_\mathrm{F}/\Delta_0)}. \qquad (2)$$

Here $\hat{\boldsymbol{\sigma}}$ is a vector of Pauli spin matrices. The spectrum of Eq. (2) gives the two split near-zero states in the entanglement spectrum. The structure of this spectrum is similar to that for boundary states [11].

Similarly, in the two-dimensional planar phase [12, 13] the low-lying part of the entanglement spectrum is given by

$$\epsilon(k_y) \approx \frac{\pi}{k_\mathrm{F} \ln(E_\mathrm{F}/\Delta_0)} k_y \hat{\sigma}_z. \qquad (3)$$

This spectrum is in correspondence with the edge-mode spectrum in the planar phase [10].

In conclusion, we found explicit expressions for the wave functions of the low-lying eigenstates of the entanglement Hamiltonian in such topological superfluids as superfluid $^3$He-B as well as a two-dimensional planar phase. We demonstrated Majorana zero modes of the entanglement Hamiltonian and found the low-lying part of the entanglement spectrum to be of relativistic type. These explicit results are in agreement with the Haldane conjecture [4] and may be useful for further analysis.

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