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Exact computation of the Special geometry for Calabi–Yau hypersurfaces of Fermat type

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When the Superstring theory is compactified on a Calabi–Yau (CY) threefold X, the low-energy effective theory is defined in terms of the Special Kähler geometry of the CY moduli space.

The Kähler metric on the moduli space is also called as the Weil–Petersson metric or tt^* metric [1] and is closely related to the Zamolodchikov metric. Apart from its own interest, the knowledge of its explicit form is useful in various contexts. In particular it enters the vacua equation in the moduli stabilization problem [2] and the holomorphic anomaly equation [3] for the higher genus B-model and allows computing distances in the moduli space. For instance, it was recently used to check the Refined Swampland Distance Conjecture in [4, 5].

We lately suggested a new much simpler method for computing the Kähler metric for a large class of CY defined as hypersurfaces in weighted projective spaces [6–8]. This method uses the correspondence between the middle cohomology of CY manifolds and the invariant Frobenius algebra associated with the potential W defining the given CY manifold. This correspondence is realized by the oscillatory integral representation for the periods of the holomorphic CY 3-form. Having established this correspondence we obtain an efficient method for computing the special geometry on the moduli space. The important feature of this method is that it allows to compute the Kähler potential for all polynomial deformations of the CY manifold as opposed to a few moduli cases before.

If a CY manifold X is realized by a quasi-homogeneous polynomial $W(x)$ in a weighted projective space $\mathbb{P}^4_{k_1,\ldots,k_5}$, then a subgroup of the cohomology group $H^3(X)$ with its natural geometric structures is isomorphic to the invariant Milnor ring R^Q defined by $W(x)$ with its special structures. From this fact, we obtain the formula for the Kähler potential $K(\phi)$

$$e^{-K(\phi)} = \sigma^+_\mu(\phi)\,\eta_{\mu\lambda}\,M_{\lambda\nu}\,\overline{\sigma^-_\nu(\phi)}, \qquad (1)$$

where $\sigma^\pm_\mu(\phi)$ are periods computed as oscillatory integrals, $\eta_{\mu\nu}$ is a natural pairing in the Milnor ring and $M_{\mu\nu}$ is the antiholomorphic involution (complex conjugation) of the ring R^Q. The three ingredients $\sigma_\mu(\phi)$, $\eta_{\mu\nu}$ and $M_{\mu\nu}$ can all be efficiently computed.

For CY manifolds defined as a Fermat hypersurface we explicitly compute the Kähler potential as a function of *all polynomial deformation parameters*. This is the main result of our paper. We can compute the real structure matrix M using decomposition of the oscillatory integrals into the product of one-dimensional ones.

Fermat CY hypersurfaces are defined as $X = \{x_1,\ldots,x_5 \in \mathbb{P}^4_{(k_1,\ldots,k_5)} |\, W(x,\phi)=0\}$,

$$W(x,\phi) = \sum_{i=1}^5 x_i^{\frac{d}{k_i}} + \sum_{\mathbf{s}=\mathbf{1}}^{h_{21}^{\text{poly}}} \phi_{\mathbf{s}} e_s(x), \quad d = \sum_{i=1}^5 k_i,$$

where $\frac{d}{k_i}$ are positive integers. The monomials $e_s(x) = e_{(s_1,\cdots,s_5)}(x) := \prod_i x_i^{s_i}$ correspond to deformations of the complex structure of X. Their weights are equal to d, $\sum_{i=1}^5 k_i s_i = d$, and each variable x_i has a nonnegative integer power $s_i \le \frac{d}{k_i} - 2$. The number of such monomials is denoted h_{21}^{poly} and is less than or equal to the Hodge number h_{21}, and $\dim H_3(X) = 2h_{21}+2$ (which can be verifed from the combinatorics of the corresponding weighted projective space).

We computed the explicit expressions for the periods σ_μ, the pairing $\eta_{\mu\nu}$, and the anti-involution M and substituted it in the above expression for the Kähler potential with the result

$$e^{-K(\phi)} = \sum_\mu (-1)^{\deg(\mu)/d} \prod_i \gamma\left(\frac{k_i(\mu_i+1)}{d}\right) |\sigma_\mu(\phi)|^2, \qquad (2)$$

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where index μ in the summation runs through the whole invariant Milnor ring basis $0 \leq \mu_i \leq \frac{d}{k_i} - 2$, $\sum_{i=1}^{5} \mu_i = 0, d, 2d, 3d$,

$$\sigma_\mu(\phi) = \sum_{n_1,\ldots,n_5 \geq 0} \prod_{i=1}^{5} \frac{\Gamma(\frac{k_i(\mu_i+1)}{d} + n_i)}{\Gamma(\frac{k_i(\mu_i+1)}{d})} \sum_{m \in \Sigma_n} \prod_s \frac{\phi_s^{m_{\mathbf{s}}}}{m_s!}, \tag{3}$$

and

$$\gamma(x) = \frac{\Gamma(x)}{\Gamma(1-x)}, \qquad \Sigma_n = \{m_{\mathbf{s}} \mid \sum_s m_{\mathbf{s}} s_i = \mu_i + \frac{d}{k_i} n_i\}. \tag{4}$$

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