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A note on reflection positivity in nonlocal gravity

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Reflection positivity. Reflection positivity is one of the axioms laid out by Osterwalder and Schrader in their seminal works [1, 2], and an important test for any attempt to quantize Einstein's theory in the quantum field theory framework. It is a necessary and sufficient condition in order for Euclidean Green's functions to (uniquely) define a Wightman quantum field theory in Minkowski space. Recently, concerns were raised [3] that weakly nonlocal theories fail to pass the basic test of perturbative reflection positivity in the coincidence limit. In this note we show that this issue does not arise for massless scalar field theories.

A Euclidean quantum field theory is said to satisfy reflection positivity when the following statement holds. For any functional $\mathcal{F}[\phi]$ of the fields whose support includes only points that have positive Euclidean time $\tau > 0$, we have

$$\langle (\Theta\mathcal{F}[\phi])\mathcal{F}[\phi]\rangle \geqslant 0, \tag{1}$$

where $\Theta\mathcal{F}[\phi]$ denotes complex conjugation and reflection with respect to the $\tau = 0$ (hyper) plane.

Considering only one "charge" [4] at fixed position X is physically equivalent to studying the properties of the propagator in the coincidence limit. Then, a simple necessary (but not sufficient) condition for perturbative reflection positivity is

$$G(\theta X - X) \geq 0. \tag{2}$$

This is in fact equivalent to the requirement that the propagator itself is positive for any vector Y

$$G(Y) \geq 0 \quad \forall\, Y. \tag{3}$$

Propagator. We simplify the expression for the propagator $G(x)$ for a general form factor $F \equiv \exp -H$ using similar manipulations as in [5], to arrive at

$$G(|x|) = \frac{2^{1-D}}{\pi^{D/2}} \int_0^\infty \mathrm{d}u\, u^{D-3}\, F(u^2\,\sigma) \\ {}_0\tilde{F}_1\left[\frac{D}{2}; -\frac{1}{4}u^2|x|^2\right]. \tag{4}$$

where we have introduced the radial coordinate in momentum space $u = \sqrt{k^2}$, and ${}_0\tilde{F}_1(a;z) = {}_0F_1(a;z)/\Gamma(a)$ is the regularized confluent hypergeometric function.

From the form (4) of the propagator we can prove the following statement. *For any form factor $F(u)$ that is a bounded positive monotonically decreasing function of the non-negative variable u, the propagator $G(Y)$ is positive for any Euclidean vector Y.* This is our main technical result. It follows from the following integral inequality [6]. For any bounded positive real function $F(u)$ that is monotonically decreasing on the positive real axis (that is, $F' < 0$ and $0 < F < \infty$ for $u \in (0,\infty)$), and for any $J(u)$ that satisfies

$$\int_0^u \mathrm{d}t\, J(t) > 0, \quad \forall u \in (0,\infty), \tag{5}$$

we have that

$$\int_0^\infty \mathrm{d}u\, F(u)J(u) > 0. \tag{6}$$

Restricting to four dimensions, we show that a sufficient condition for perturbative reflection positivity in the coincidence limit is simply that the *form factor F* is a monotonically decreasing function

$$F' < 0. \tag{7}$$

Theories concerned. Two popular classes of theories concerned by the above are those with form factors $F \equiv e^{-H(\sigma\,\Box)}$ such that

$$H_K(z) = \alpha\left[\log(z) + \Gamma(0,z) + \gamma_E\right], \ \operatorname{Re} z > 0, \tag{8}$$

$$H_T(p) = \frac{1}{2}\left[\log\left(p^2\right) + \Gamma\left(0,p^2\right) + \gamma_E\right], \ \operatorname{Re} p^2 > 0. \tag{9}$$

Here, α is an integer, p is a polynomial of degree n in the variable $z \equiv \sigma\Box$ and γ_E is the Euler–Mascheroni con-

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stant. The parameters α and n must satisfy the super-renormalizability conditions

$$\alpha > D-1, \quad n > D-1, \tag{10}$$

respectively. We also consider form factors that are asymptotically exponential and are of the form, see Fig. 1

$$F = e^{-H} = e^{-(-\sigma\Box)^n}, \; n \in \mathbb{N}^+. \tag{11}$$

Such form factors have been studied in the context of string theory and gauge theories.

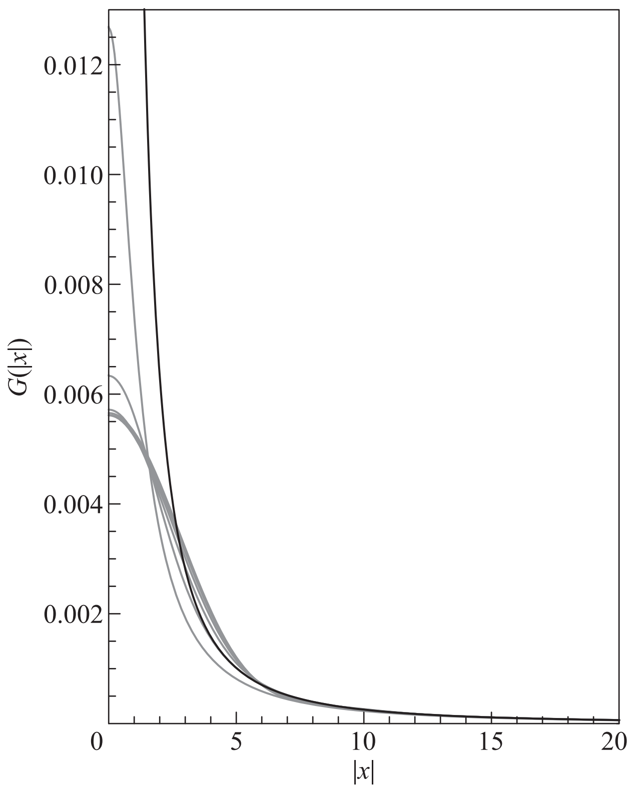

Fig. 1. The propagators for the form factor of Eq. (11) for $D = 4$ and $n = 1/2, 1, 3/2, 2, 5/2, 3$ (gray). The propagator for General Relativity is drawn for comparison (black). The propagators for the non-local theory are positive and well behaved in the ultraviolet, including the coincidence limit $|x| = 0$. The infrared behaviour is identical as that of General Relativity. We note that as n increases, $G(0)$ converges to the value $\sim 6 \cdot 10^{-3}$. The nonlocality scale σ has been set to unit here

All form factors of the form (11) satisfy the above. This result shows that the violation of perturbative reflection positivity noticed in [3], where such form factors and the test (2) were considered for massive scalar field theories, does not arise for the massless case. The reason that the propagator can be negative for the massive case is that when $m \neq 0$, there is an extra factor $\frac{u^2}{u^2+m^2}$ in the integral, which is equal to unit when $m = 1$. This function is monotonically *increasing* in the interval $u \in (0, \infty)$.

Returning to the massless case, for form factors as in Eq. (9) we have that

$$\frac{\mathrm{d}F}{\mathrm{d}u} = -\frac{\mathrm{d}p}{\mathrm{d}u}\frac{\mathrm{d}H}{\mathrm{d}p}F. \tag{12}$$

Since

$$\frac{\mathrm{d}H}{\mathrm{d}p} = \frac{\alpha}{p}\left(1 - e^{-p^2}\right), \tag{13}$$

the inequality (7), and thus also (3), is automatically satisfied whenever p is a monomial, which includes the case of Eq. (8). Then, (3) and by extension (2) is shown to hold for these cases as well.

Conclusions. We gave a simple proof for the positivity of the propagator for a large class of massless nonlocal theories in four dimensions. As a consequence, a basic test for unitarity, perturbative reflection positivity in the coincidence limit, was shown to hold. This result covers theories defined by exponential form factors as well as form factors that have the properties required of candidates for a well defined quantum gauge or gravitational theory. Furthermore, we have provided numerical evidence that the same is true also for higher dimensions and for nonlocal theories not covered by the proof given here. Our results show that the results in [3] do not imply a generic violation of perturbative reflection positivity in the coincidence limit for weakly nonlocal field theories, including quantum gravity or gauge theories. We have shown here that this issue does not arise for the massless case.

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