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Amplitude of waves in the Kelvin-wave cascade

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In quantum turbulence, velocity fluctuations and vortex reconnections drive oscillating motion of quantized vortices – Kelvin waves [1]. Kelvin waves interact non-linearly and support a cascade of energy towards smaller length scales and larger wave numbers [2]. The theory of the Kelvin-wave cascade was the subject of controversy, until finally the L'vov–Nazarenko model [3, 4] got supported by numerical simulations [5, 6]. Recently, progress in experimental techniques [7–9] enables controllable excitation of waves on nearly straight vortices and potential observation of the Kelvin-wave cascade. In this work we provide relation of the energy flux carried by the cascade to the amplitude of the excited Kelvin waves, which is important for analysis of such experiments.

We assume that the Kelvin-wave cascade on a vortex of length L (cm) carries the energy flux $\tilde{\epsilon}$ (erg/s) and starts from the wave number $k_{\rm min}$ (cm^{-1}). Our goal is to find the amplitude A_k (cm) of the Kelvin wave with the wave number k (cm^{-1}). We start by noting that in the local induction approximation the energy of a vortex line $E_{\rm v}$ is given by the product of its length L and the vortex tension $\nu_{\rm s}$

$$E_{\rm v} = \nu_{\rm s} L, \quad \nu_{\rm s} = \rho_{\rm s}\frac{\kappa^2\Lambda}{4\pi}, \quad \Lambda = \ln\left(\frac{\ell}{a_0}\right). \qquad (1)$$

Here $\rho_{\rm s}$ is the superfluid density, κ is the circulation quantum, a_0 is the vortex core radius and ℓ is the mean intervortex spacing or the size of the enclosing volume, in the case of a single vortex. For a spiral Kelvin wave of the radius A_k and wavelength $\lambda_k = 2\pi/k$, the increase of the length compared to that of the straight vortex is

$$L_k = \left(\sqrt{\lambda_k^2 + (2\pi A_k)^2} - \lambda_k\right)\frac{L}{\lambda_k} \approx L\,\frac{2\pi^2 A_k^2}{\lambda_k^2}, \qquad (2)$$

where we assumed that $A_k \ll \lambda_k$. Thus the total energy due to Kelvin waves is

$$E_{\rm kw} = \sum_{k=\pm k_{\rm min}}^{\pm\infty} \nu_{\rm s} L_k = L\sum_{k=k_{\rm min}}^{\infty} \nu_{\rm s} A_k^2 k^2 = = L\frac{\nu_{\rm s}}{k_{\rm min}}\int_{k_{\rm min}}^{\infty} A_k^2 k^2\,dk. \qquad (3)$$

Comparing this result to the expression of the energy via the Kelvin-wave frequency ω_k and the combined occupation number N_k for modes with $\pm k$ [4]

$$E_{\rm kw} = \rho_{\rm s} L\int_{k_{\rm min}}^{\infty} E_k\,dk, \quad E_k = \omega_k N_k, \quad \omega_k = \frac{\kappa\Lambda}{4\pi}k^2, \qquad (4)$$

we find

$$A_k^2 = \frac{k_{\rm min}}{\kappa} N_k. \qquad (5)$$

The L'vov–Nazarenko spectrum is [4]

$$E_k = C_{\rm LN}\frac{\kappa\Lambda\epsilon^{1/3}}{\Psi^{2/3}k^{5/3}}, \qquad C_{\rm LN} \approx 0.304, \qquad (6a)$$

$$\Psi = \frac{8\pi}{\Lambda\kappa^2}\int_{k_{\rm min}}^{\infty} E_k dk. \qquad (6b)$$

Here ϵ is the energy flux per unit length and per unit mass. It is related to the flux $\tilde{\epsilon}$ as

$$\epsilon = \frac{\tilde{\epsilon}}{L\rho_{\rm s}}, \qquad [\epsilon] = \frac{\rm cm^4}{\rm s^3}. \qquad (7)$$

Solving Eq. (6) for Ψ we get

$$\Psi = \frac{(12\pi C_{\rm LN})^{3/5}\epsilon^{1/5}}{\kappa^{3/5}k_{\rm min}^{2/5}} \qquad (8)$$

and from Eq. (5) finally

$$A_k^2 = 2\left(\frac{2\pi^3 C_{\rm LN}^3}{9}\right)^{1/5}\frac{k_{\rm min}^{19/15}\epsilon^{1/5}}{\kappa^{3/5}k^{11/3}} \approx \approx 1.4\,\frac{k_{\rm min}^{19/15}}{\kappa^{3/5}k^{11/3}}\left(\frac{\tilde{\epsilon}}{L\rho_{\rm s}}\right)^{1/5}. \qquad (9)$$

Note that $A_k \propto \tilde{\epsilon}^{1/10}$. Thus determination of the amplitude from the energy flux should be relatively reliable,

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while the reverse procedure is bound to be very uncertain.

The total increase of the vortex line length due to Kelvin waves can be found from the energy as $L_{\rm kw} = E_{\rm kw}/\nu_{\rm s}$, where $E_{\rm kw}$ is given by Eqs. (4), (6a) and (8):

$$L_{\rm kw} = \frac{E_{\rm kw}}{\nu_{\rm s}} = L\,\frac{2^{1/5}(3\pi C_{\rm LN})^{3/5}\epsilon^{1/5}}{\kappa^{3/5}k_{\rm min}^{2/5}}. \tag{10}$$

Thus for the relative increase we get a simple formula

$$\frac{L_{\rm kw}}{L} = \frac{E_{\rm kw}}{E_{\rm v}} = \frac{\Psi}{2}. \tag{11}$$

In cases, where instead of a single vortex, one considers a vortex array with the total length L occupying volume V with the density $\mathcal{L} = L/V = \ell^{-2}$, it might be more convenient to operate with the standard 3-dimensional energy flux $\varepsilon = \epsilon\mathcal{L}$ per unit mass and unit volume, $[\varepsilon] = \mathrm{cm}^2\,\mathrm{s}^{-3}$. Then for the increase $\mathcal{L}_{\rm kw}$ of the vortex-line density due to Kelvin waves, we find using Eqs. (8) and (11)

$$\frac{\mathcal{L}_{\rm kw}}{\mathcal{L}} = \frac{\Psi}{2} = \left[\frac{2(3\pi C_{\rm LN})^3\varepsilon}{b^2\mathcal{L}^2\kappa^3}\right]^{1/5} \approx 2.2\left(\frac{\varepsilon}{b^2\mathcal{L}^2\kappa^3}\right)^{1/5}, \tag{12}$$

where we introduced

$$b = k_{\rm min}\ell \sim 1. \tag{13}$$

We note that the numerical value of the prefactor in Eqs. (9) and (12) should be taken with caution. In the calculations we assume that the total energy of Kelvin waves can be found by the integral (4) limited from below by $k_{\rm min}$ with the scale-invariant spectrum (6). In reality this spectrum was derived for $k \gg k_{\rm min}$ while the main contribution to $E_{\rm kw}$ is coming from the region $k \simeq k_{\rm min}$. Behavior of the Kelvin-wave spectrum in this long-wavelengths region may be different and, in general, is not universal.

In some applications, the tilt θ of a vortex carrying Kelvin waves with respect to the direction of the straight vortex is of interest. The averaged tilt angle can be connected to the length increase

$$L_{\rm kw} = \int_0^L \sqrt{1+\tan^2\theta(z)}\,\mathrm{d}z - L \simeq \frac{1}{2}\langle\tan^2\theta(z)\rangle L. \tag{14}$$

Together with Eq. (11) this results in

$$\langle\tan^2\theta(z)\rangle \simeq 2\,\frac{L_{\rm kw}}{L} = \Psi, \tag{15}$$

where Ψ is given by Eq. (8).

To conclude, we have found the dependence of the amplitude of the Kelvin waves, of the length increase of the vortex, and of the average vortex tilt on the energy flux carried by the Kelvin-wave cascade. The results are applicable in the regime of weak turbulence of Kelvin waves, which is uniform along the vortex. We stress that the amplitude of the Kelvin waves, generated when a vortex is mechanically agitated, does not necessary coincide with the amplitude of the motion of the agitator. Solving the problem of excitation of Kelvin waves in a realistic experimental geometry remains a task for future research.

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