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Generalized unimodular gravity in Friedmann and Kantowski–Sachs universes

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One of the oldest modified theories of gravity is unimodular gravity, dating back to the paper by Einstein [1]. The recent rebirth of this idea is connected with papers [2, 3]. The main point of unimodular gravity consists of the fact that when one requires that the determinant of the metric is fixed, the cosmological constant arises as an integration constant in the Einstein equations. The unimodular gravity theories can essentially be generalized by using the Arnowitt—Deser–Misner (ADM) [4] approach to gravity.

Such a generalization was suggested recently in paper [5]. If one treats the lapse function N not as Lagrange multiplier, giving one of the constraints of the theory, but as a given function of the determinant of the spatial metric γ, then in the equations of motion an effective matter arises with the equation of state parameter w given by $w = 2\frac{d\ln N(\gamma)}{d\ln\gamma}$.

Thus, on treating one of the Lagrange multipliers of the General Relativity, i.e., the lapse function N not as a Lagrange multiplier, but as a given function of other variables, we freeze one of the symmetries of the system and as a result the effective matter content of the theory becomes richer. This phenomenon is quite well-known and was pioneered by Dirac in paper [6] dedicated to electrodynamics.

In spite of its simplicity the model of generalized unimodular gravity [5] imposes some interesting problems and opens some attractive prospects due to its unexpected flexibility. In paper [7] the Hamiltonian formalism for this model, treated as a rather complicated example of a constrained dynamical system [8], was considered in detail. Especially interesting in this context is the question of the determination of the number and the character of the physical degrees of freedom, arising here. The paper [9] was devoted to the inflationary model based on generalized unimodular gravity and the behaviour of linear perturbations in this model was studied.

However, the model [5] opens some interesting opportunities already at the level of a simple minisuperspace models with finite number of degrees of freedom. We shall discuss here some of them. For a flat Friedmann model with the metric $ds^2 = -N^2(t)dt^2 + a^2(t)dl^2$, $\gamma = a^6$ and the equation of state is simply $w = \frac{1}{3}\frac{d\ln N(a)}{d\ln a}$. One can derive this equation directly from the Friedmann model. The Lagrangian for the flat Friedmann universe without matter can be written as $L = \frac{\dot{a}^2 a}{N}$. If we now treat the lapse function as a function of the scale factor a, the variation with respect to a gives the following Euler–Lagrange equation: $2\frac{\ddot{a}a}{N} + \dot{a}^2\frac{d(a/N)}{da} = 0$, where the "dot" signifies the differentiation with respect to the time parameter t. This equation has the first integral $\frac{\dot{a}^2 a}{N} = C$, where C is a constant. Dividing this equation by Na^3, we obtain

$$\frac{\dot{a}^2}{N^2a^2} = \frac{1}{a^2}\left(\frac{da}{d\tau}\right)^2 = \frac{C}{Na^3}, \tag{1}$$

where τ is the cosmic or synchronous time $d\tau = Ndt$. This equation can be interpreted as the first Friedmann equation for a flat universe filled with matter having the energy density $\varepsilon = \frac{C}{Na^3}$. On remembering the energy conservation law

$$\frac{d\varepsilon}{da} = -3\frac{\varepsilon + p}{a}, \tag{2}$$

we can immediately find the pressure

$$p = -\frac{1}{3}a\frac{d\varepsilon}{da} - \varepsilon = \frac{C}{3N^2a^2}\frac{dN}{da} = \frac{1}{3}\frac{d\ln N}{d\ln a}\varepsilon,$$

which confirms the relation presented above.

It is known that the observed cosmic acceleration of the universe requires the presence of a so called dark

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energy with negative pressure. Some observations indicate that the corresponding equation of state parameter is less than -1: $w < -1$. Such a kind of dark energy is called "phantom dark energy". The evolution in the presence of such energy implies the future encounter with a cosmological singularity called "Big Rip" [10, 11]. Its scale factor and its time derivative tend to infinity.

However, one can imagine a less dramatic scenario for the development of the universe, wherein the phantom or super-acceleration stage is a temporary one. In this case the universe should pass through the phantom divide line which means that the sign of the expression $w + 1$ changes. We wish to show that, at least at the level of the Friedmann model, the generalized unimodular gravity can easily describe the phantom divide line crossing.

Indeed, it is enough to choose the lapse function as follows:

$$N = \frac{D}{a^5} + Fa. \tag{3}$$

On remaining in the field of minisuperspace models with a finite number of degrees of freedom, we can already suggest a further simple generalization of unimodular gravity. In particular the lapse function can depend not on the determinant of the spatial metric, but on some other combination of components of the spatial part of the metric. Let us consider, for example, a hyperbolic Kantowski–Sachs universe [12] with the metric

$$ds^2 = N^2(t)dt^2 - b^2(t)dr^2 - a^2(t)(d\chi^2 + \sinh^2\chi d\phi^2). \tag{4}$$

If we fix the time parameter by choosing the lapse function as $N = a$, we can find the metric of the Kantowski–Sachs universe in an explicit form. One of the possible solutions is $a(t) = a_0 \cosh^2 \frac{t}{2}$; $b = b_0 \tanh \frac{t}{2}$. It is interesting to note that there is a duality between the Kantowski–Sachs cosmological solutions and the static spherically symmetric solutions. This duality was found in paper [13] and further investigated in [14]. If we exchange the variables t and r and then make the substitution $\chi \to i\theta$, we obtain the following metric [13]:

$$ds^2 = b_0^2 \tanh^2 \frac{r}{2} dt^2 - a_0^2 \cosh^4 \frac{r}{2}(dr^2 + d\theta^2 + \sin^2\theta d\phi^2). \tag{5}$$

On introducing a new variable $R \equiv a_0 \cosh^2 \frac{r}{2}$, we can rewrite the metric obtained in the standard Schwarzschild form. Let us now suppose that, at the beginning, we had fixed $N = a$. In this case the lapse function is not a function of the determinant of the spatial metric, thus we are considering a further generalization of unimodular gravity. We obtain a new first integral of the equations of motion: $\frac{\dot{a}^2 b}{N} + \frac{2\dot{a}\dot{b}a}{N} - Nb = A$. The expression for a is now the same as before, while the scale factor b is $b = b_0 \tanh \frac{t}{2} - \frac{A}{a_0}$. On using the duality relations, we obtain the following Schwarzschild-type metric:

$$ds^2 = \left[b_0^2\left(1 - \frac{a_0}{R}\right) - 2\frac{Ab_0}{a_0}\sqrt{1 - \frac{a_0}{R}} + \frac{A^2}{a_0^2}\right]dt^2 - \\ - \frac{dR^2}{1 - \frac{a_0}{R}} - R^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{6}$$

We see that while the spatial part of the metric has not changed, the coefficient g_{00} for dt^2 has changed essentially. If the constant A was positive then the metric coefficient vanishes at

$$R_0 = \frac{a_0}{1 - \frac{A^2}{a_0^2 b_0^2}} > a_0,$$

provided $A^2 < a_0^2 b_0^2$. We should then think of how to describe the continuation of the metric into the region where $R < R_0$ and then to $R < a_0$. If A is negative (the energy density of the effective matter is negative) the expression for b cannot become equal to zero, but we still stumble upon the problem of its behaviour for $R < a_0$.

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