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Classification of emergent Weyl spinors in multi-fermion systems

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Electrons in solids are described by multi-component spinors carrying band index. However, at low energies we may describe electrons by effective spinors with the essentially reduced number of components. Only those energy bands are relevant that cross Fermi energy. In Dirac/Weyl semimetals with Fermi points the emergent spinors are two component if Fermi energy is close to the position of the Fermi points. Due to the repulsion of energy levels the Fermi points are unstable unless they are protected by topology. Therefore, the effective description in terms of the two-component spinors typically survives in the case when the topological invariants protecting the Fermi points are nonzero.

In the present paper we extend the existing classification [1] of Weyl points into two directions. First of all, we notice that in general case of interacting systems the two topological invariants N_3 and $N_3^{(3)}$ may be introduced. Both are composed of the Green functions. They become different in general case. As a result in case of the minimal values $N_3, N_3^{(3)} = \pm 1$ we obtain the four topologically distinct types of Weyl fermions. We call two of them the left-handed and the right-handed particles, and the other two – the left-handed and the right-handed "anti-particles". The latter types of Weyl points are referred to as the anti-Weyl points, they may appear only in the presence of interactions. This classification extends the conventional one, which considers the two types of relativistic Weyl fermions – the left-handed, and the right-handed. Another direction for the extension of the classification of emergent Weyl fermions is related to consideration of the non-homogeneous systems following methodology of [2].

We start from the consideration of equilibrium condensed matter system with the n-component spinors ψ at zero temperature. Let us consider the Green functon

$$\langle x_1, t_1|\hat{G}|x_2, t_2\rangle = \frac{-i}{Z}\int D\psi D\bar{\psi} D\Phi \exp\Big(iR[\Phi] + \\ + i\int dt \sum_{\mathbf{x}} \bar{\psi}_{\mathbf{x}}(t)(i\partial_t - \hat{H}(\Phi)e^{-i\epsilon})\psi_{\mathbf{x}}(t)\Big)\psi_{x_1}(t_1)\bar{\psi}_{x_2}(t_2). \quad (1)$$

Here hamiltonian $\hat{H}(\Phi)$ is a Hermitian operator depending on the field Φ, which provides interactions, $R[\Phi]$ is its action. Sum over points of coordinate space is to be understood as an integral over d^3x for continuous coordinate space. However, we may also consider lattice tight-binding models, in which case we have the sum over the lattice points. Factor $e^{-i\epsilon}$ is introduced here in order to point out how the poles in Feynmann diagrams are to be treated. ϵ is assumed to be very small.

Several branches of spectrum of $\hat{H}$ repel each other. Therefore, the minimal number $n_{\rm reduced} = 2$ of branches are able to cross each other. And this minimal number is fixed by topology of momentum space. Let us consider the position of the crossing of 2 branches of $\hat{H}$. At low energies (close to the Fermi level coinciding with the branch crossing point) the contribution to the physical observables of the reduced two-component fermions Ψ, $\bar{\Psi}$ dominates over the contribution of the gapped ones.

In the homogeneous case the Green function may be written as $\langle x_1, t_1|\hat{G}|x_2, t_2\rangle = G(t_1 - t_2, x - y)$. In order to construct the topological invariants responsible for the stability of Fermi points we may also use the Green function of the reduced low energy theory

$$G(t_1 - t_2, x - y) = \frac{-i}{Z}\int D\Psi D\bar{\Psi} D\Phi \times \\ \times \exp\Big(iR[\Phi]\Big)\exp\Big(i\int dt \sum_{\mathbf{x}} \bar{\Psi}_{\mathbf{x}}(t)(im'_{\Phi,a}\sigma^a\partial_t + \\ + [\mu m'_{\Phi,a}\sigma^a - m^L_{\Phi,k}(\hat{\mathbf{p}})\hat{\sigma}^k - m_\Phi(\hat{\mathbf{p}})]e^{-i\epsilon})\Psi_{\mathbf{x}}(t)\Big) \times \\ \times \Psi_x(t_1)\bar{\Psi}_y(t_2). \quad (2)$$

Here $\mathbf{p}$ is momentum, σ^a are Pauli matrices, while m^L, m, and m' are the functions of momenta depending also on the field Φ. The latter is assumed to be slowly varying, so that the commutators $[\hat{\mathbf{p}}, \Phi]$ may

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be neglected. Then we compose the Fourier transform $G(P_0, P_1, P_2, P_3)$ of the Green function. We substitute to $G(P_0, P_1, P_2, P_3)$ the values $P_0 = i\omega$ and $P_j = p_j$ for $j = 1, 2, 3$. The first topological invariant is given by

$$N_3 = \frac{1}{3!\,4\pi^2}\int_\Sigma \mathrm{Tr}\Big[G(\omega,\mathbf{p})dG^{-1}(\omega,\mathbf{p})\wedge \wedge\, dG_H(\omega,\mathbf{p})\wedge dG_H^{-1}(\omega,\mathbf{p})\Big]. \quad (3)$$

Here Σ is the three-dimensional hypersurface surrounding the Fermi point in $4D$ momentum space (composed of $3D$ momentum space and the axis of $\omega = p_4$).

In addition, we define

$$G_H(\omega,\mathbf{p}) = \frac{1}{i\omega - G(0,\mathbf{p})^{-1}}. \quad (4)$$

The second topological invariant is defined as

$$N_3^{(3)} = \frac{1}{3!\,4\pi^2}\int_\Sigma \mathrm{Tr}\Big[G_H(\omega,\mathbf{p})dG_H^{-1}(\omega,\mathbf{p})\wedge \wedge\, dG_H(\omega,\mathbf{p})\wedge dG_H^{-1}(\omega,\mathbf{p})\Big]. \quad (5)$$

For the minimal values of N_3 and $N_3^{(3)}$ we classify the Weyl points according to the following Table 1.

Table 1. Weyl fermions and values of topological invariants

Fermion type	N_3	$N_3^{(3)}$
Left-handed Weyl point	+1	+1
Right-handed Weyl point	-1	-1
Left-handed anti-Weyl point	+1	-1
Right-handed anti-Weyl point	-1	+1

In the systems with weak inhomogeneity as above one may define the two topological invariants. We define Wigner transformation of Green function $G_W^{(M)}(p,x) = \int d^4\langle x + r/2|\hat{G}|x - r/2\rangle e^{i(p_0 r_0 - \mathbf{pr})}$. Wick rotation we introduce $p_0 = i\omega = iP_4$, and $P_j = p_j$ for $j = 1, 2, 3$; $x_0 = -iX_4$, $X_j = x_j$, and denote the Euclidean Wigner transformation of Green function: $G_W(P,X) = G_W^{(M)}(p,x)$. We also define Q_W that obeys $Q_W * G_W = 1$. Here by $*$ we denote the Moyal product

$$* = e^{\frac{i}{2}(\overleftarrow{\partial_{X^i}}\overrightarrow{\partial_{P_i}} - \overleftarrow{\partial_{P_i}}\overrightarrow{\partial_{X^i}})}.$$

The first topological invariant is given by

$$N_3 = \frac{1}{3!\,4\pi^2\,|\mathbf{V}|}\int_\Sigma \int d^3X\, \mathrm{Tr}\Big[G_W(P,X) * dQ_W(P,X) * \wedge\, dG_W(P,X) * \wedge dQ_W(\omega,P,X)\Big]. \quad (6)$$

$|\mathbf{V}|$ is the three-volume of the system. Here Σ is the three-dimensional hypersurface surrounding the singularity $\mathcal{M}^{(i)}$ of expression standing inside the integral. The general procedure for the construction of such invariants has been proposed in [2]. In addition, we define

$$Q_{H,W} = i\omega - Q_W(P,X)\Big|_{\omega=0}$$

and $G_{H,W}$ obeys

$$Q_{H,W} * G_{H,W} = 1.$$

Then the second topological invariant can be defined as

$$N_3^{(3)} = \frac{1}{3!\,4\pi^2\,|\mathbf{V}|}\int_\Sigma \int d^3X\, \mathrm{Tr}\Big[G_{H,W}(P,X) * dQ_{H,W}(P,X) * \wedge dG_{H,W}(P,X) * \wedge dQ_{H,W}(\omega,P,X)\Big]. \quad (7)$$

Thus we come to an unexpected conclusion: in general case in multi-fermion systems with minimal values of $N_3, N_3^{(3)} = \pm 1$ the emergent Weyl fermions appear in four rather than two topologically different classes. The appearance of anti-Weyl points with $N_3 = -N_3^{(3)}$ has sense only in the presence of the conventional Weyl fermions with $N_3 = N_3^{(3)}$. Without the latter R transformation $\Psi \to -\Psi, \bar{\Psi} \to \bar{\Psi}$ brings Weyl fermions to the type of "particles" with $N_3 = N_3^{(3)}$. At the same time if both types of Weyl fermions co-exist, the interesting phenomena may occur. For example, a couple of Weyl fermions $(N_3, N_3^{(3)}) = (-1,-1)$ and $(-1,+1)$ may merge giving marginal Weyl point with $(N_3, N_3^{(3)}) = (-2, 0)$. We gave an example of the lattice condensed matter system, in which two right-handed Weyl points exist. One of them is of the type of a Weyl point while another one is of the type of the anti-Weyl point. Changing smoothly parameter α of the system it is possible to bring it to the state, in which the Weyl point and the anti-Weyl point merge giving the marginal Weyl point with $N_3 = -2$ and $N_3^{(3)} = 0$.

We would like to notice that the classification presented here may be relevant for the high energy physics and applications of quantum field theory to cosmology (see [3, 4] and references therein). Then the appearance of the four (rather than two) topologically distinct types of Weyl fermions may be assumed from the very beginning. Such a construction may be relevant for the proper theory of quantum gravity, which should include the strong fluctuations of vierbein giving rise to all four types of the Weyl points [4].

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1. G. E. Volovik, *The Universe in a Helium Droplet*, Clarendon Press, Oxford (2003).
2. M. A. Zubkov and X. Wu, Ann. Phys. **418**, 168179 (2020); arXiv:1901.06661.
3. G. E. Volovik, arXiv:2011.06466.
4. S. N. Vergeles, arXiv:1903.09957.