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REMARK ON EQUIVALENCE OF TOPOLOGICAL AND QUANTUM 2D GRAVITY

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We demonstrate the equivalence of Virasoro constraints imposed on continuum limit of partition function of Hermitean 1-matrix mod·1 and the Ward identities of Kontsevich's model.

1. From the early days of matrix models there is a belief that partition functions of two *a priori* different models of 2d gravity should coincide. The first is the (square root of) partition function $\sqrt{Z_{qg}\{T_n\}}$ of ordinary Polyakov's quantum 2d gravity, described by the continuum limit of Hermitean 1-matrix model [1] (and is known to be a τ-function of KdV hierarchy [2]). The second partition function Z_{tg} of Witten's topological gravity [3] is *a priori* defined as a generating functional of intersection indices of divisors on module spaces of Riemann surfaces with punctures.

Later it was discovered [4,5] that $\sqrt{Z_{qg}\{T_n\}}$ satisfies a set of Virasoro constraints:

$$\mathcal{L}_n\sqrt{Z_{qg}\{T\}} = 0,\ n \geq -1 \tag{1}$$

$$\mathcal{L}_n = \sum_{k\geq\delta_{n+1,0}} T_k\frac{\partial}{\partial T_{k+n}} + \sum_{\substack{a+b=n1\\ a,b0}} \frac{\partial^2}{\partial T_a\partial T_b} + \delta_{n+1,0}\cdot\frac{T_0^2}{16} + \delta_{n,0}\cdot\frac{1}{16}$$

Eq. (1) can be deduced [6] as a continuum limit of Ward identities in discrete matrix model, associated with the shift of integration variables [7]

$$X \to X + \epsilon X^{p+1} \tag{2}$$

in the integral

$$Z_N^{(d)}\{t\} = \int DX\ \exp -trV\{X\} \tag{3}$$

$$V\{X\} = \sum_{k=0}^{\infty} t_k X^k$$

However, it may be more reasonable to take these Virasoro constraints for a straightforward definition of $Z_{qg}\{T\}$, which does not refer to a sophisticated change of variables $\{t\} \to \{T\}$ [6] and to discrete model (3) at all.

As for Z_{tg} it was recently represented by M.Kontsevich [8] in terms of another matrix model

$$Z_{tg} = lim_{sizeX\to\infty} Z_{tg}^{(d)} \tag{4}$$

where

$$Z_{tg}^{(d)} = \frac{1}{C[M]} \int DX \ \exp - tr\{MX^2 + X^3\} \tag{5}$$

with

$$C[M] = \int DX \ \exp - tr\{MX^2\} = \det(M \otimes I + I \otimes M)^{-\frac{1}{2}} \tag{6}$$

and X, M being (anti)Hermitean matrices.

It is a simple combinatorial result, that as soon as the size of X goes to infinity $Z_{tg}^{(d)}$ becomes dependent only on the variables

$$T_m = \frac{3^{2m+1}}{m+\frac{1}{2}} tr \ M^{-2m-1} + \frac{4}{3\sqrt{3}} \delta_{m,1} \tag{7}$$

The explicit formulation of the original suggestion in these terms would be

$$Z_{tg}\{T_m\} = \sqrt{Z_{qg}\{T_n\}} \tag{8}$$

which in particular implies what is known as Witten's suggestion, *i.e.* that $Z_{tg}\{T\}$ is like $\sqrt{Z_{qg}\{T\}}$ a τ-function of KdV-hierarchy. A necessary condition for (8) to hold is that $Z_{tg}\{T\}$ defined by (4-6) satisfies the same Virasoro constraints (1).

This is the statement which we prove in the letter. Moreover we shall prove a relation involving $Z_{tg}^{(d)}$ (see eq.(16) below), *i.e.* valid for *finite* dimensional matrices X, which implies the entire set of Virasoro constraints only as the size of matrix goes to infinity.

This statement is equivalent to (8) modulo:

(i) the assertions which have been made by Kontsevich in the derivation of (4-6) from Penner's formalism of fat graphs, and

(ii) the so far subtle problem of uniqueness of solutions to Virasoro constraints (1).

2. Since the Ward identities are a little obscure in Kontsevich's presentation (4-6), we begin with a slight reformulation of his model.

After a shift of integration variable $X \to X - \frac{M}{3}$ and the redefinition $M = 3\Lambda^2$, we obtain

$$\mathcal{F}\{\Lambda\} \equiv \int DX \ \exp(-trX^3 + tr\Lambda X)$$
$$= C[\sqrt{\Lambda}] \exp(-\frac{2}{3\sqrt{3}} tr\Lambda^{3/2}) Z_{tg}^{(d)} \tag{9}$$

with

$$C[\sqrt{\Lambda}] = \det(\sqrt{\Lambda} \otimes I + I \otimes \sqrt{\Lambda})^{-\frac{1}{2}} \tag{10}$$

The functional $\mathcal{F}\{\Lambda\}$ can be considered as a kind of matrix-Fourier transform of exponential cubic potential trX^3. It satisfies obvious Ward identities, associated with a shifts of integration variables

$$X \to X + \epsilon_p \tag{11}$$

namely

$$tr(\epsilon_p \frac{\partial^2}{\partial \Lambda_{tr}^2} - \frac{1}{3}\epsilon_p \Lambda)\mathcal{F}\{\Lambda\} = 0 \tag{12}$$

Here ϵ_p stands for any X-independent *matrix*, wh·-h may be diagonal simultaneously with Λ , *e.g.* $\epsilon_p = \Lambda^p$.

Let us substitute in (12) the functional $\mathcal{F}$ in the form, inspired by (9)

$$\mathcal{F}\{\Lambda\} = C[\sqrt{\Lambda}] \exp(-\frac{2}{3\sqrt{3}} tr\Lambda^{3/2}) Z\{T_m\} \tag{13}$$

with

$$T_m = \frac{1}{m+\frac{1}{2}} tr\Lambda^{-m-\frac{1}{2}} + \frac{4}{3\sqrt{3}}\delta_{m,1} \tag{14}$$

Our statement is that after the substitution of (13) into (12) it turns into

$$\sum_{n\geq -1} tr(\epsilon_p \Lambda^{-n-2}) \mathcal{L}_n Z = 0 \tag{15}$$

The identity

$$\frac{1}{\mathcal{F}} tr(\epsilon_p \frac{\partial^2}{\partial \Lambda_{tr}^2} - \frac{1}{3}\epsilon_p \Lambda)\mathcal{F} = \frac{1}{Z} \sum_{n\geq -1} tr(\epsilon_p \Lambda^{-n-2}) \mathcal{L}_n Z \tag{16}$$

is valid *for any size of the matrix* Λ, but only in the limit of infinitely large Λ :

(i) it is reasonable to substitute $Z_{tg}^{(d)}$ in (9) by Z_{tg}, which depends only on $tr\Lambda^{-q}$ with half-integer $q's$;

(ii) the T_m's in (14) are really independent variables;

(iii) all the quantities

$$tr(\epsilon_p \Lambda^{-n-2}) \tag{17}$$

(*e.g.* $tr\Lambda^{p-n-2}$) become algebraically independent, so that eq. (15) implies that

$$\mathcal{L}_n Z_{tg}\{T\} = 0,\, n \geq -1 \tag{18}$$

This concludes the derivation of Virasoro constraints for $Z_{tg}\{T\}$ defined as continuum limit of Kontsevich's partition function.

Note that the fact that operators $\mathcal{L}_n$ in (15) contain only second T-derivatives, is a direct consequence of that only double Λ-derivatives arise in the *l.h.s.* and thus of the cubic nature of potential trX^3 in Kontsevich's model. It is very interesting (in particular, from the point of view of multimatrix and Potts models) to study the matrix-Fourier transform of potentials like trX^{K+1} for any degree $K+1$. Amusingly enough, the corresponding Ward identities being transformed to the form of (15) contain operators similar to the higher-spin operators of the W_K-algebra instead of the Virasoro generators $\mathcal{L}_n$.

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