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Minlos–Faddeev regularization of zero-range interactions in the three-body problem

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In studies of the universal low-energy dynamics, it is natural to use a zero-range model for short-range two-body interactions. Nevertheless, in the few-body problem this leads to essential difficulties, which manifest, e.g., in the appearance of Efimov or Thomas effects. Minlos and Faddeev suggested a modification of zero-range interactions in the influential paper [1]. A main idea was to add a regularizing term, which diminishes the interaction strength in vicinity of the triple-collision point. It was shown that the Efimov or Thomas effects are suppressed if a strength of the regularizing term σ exceeds the critical value σ_c. Later on it was declared [2–4] that $\sigma \geq \sigma_c$ is sufficient for unambiguous formulation of the three-body problem.

The present work is aimed to study the Minlos–Faddeev regularization both for the two-component system consisting of two identical bosons of mass m interacting with distinct particle of mass m_1 and for the system consisting of three identical bosons. The zero-range interaction is completely determined by the scattering length, which could be taken as a length scale, as a result, the mass ratio m/m_1 becomes a single essential parameter in the problem.

In terms of the scaled Jacobi variables $\mathbf{x}$ and $\mathbf{y}$, the Hamiltonian in the center-of-mass frame is formally defined as the six-dimensional Laplace operator supplemented by the boundary conditions at zero distance between the interacting particles,

$$\lim_{x\to 0}\left[\frac{\partial \log(x\Psi)}{\partial x} - \frac{\sigma}{\cos\omega}\frac{\theta(y)}{y}\right] = -\mathrm{sign}(a), \qquad (1)$$

where $\theta(y)$ is an arbitrary bounded function normalized by $\theta(0) = 1$ and $\theta'(0) = 0$. The factor $1/\cos\omega$ is introduced for convenience and the kinematic angle ω is defined by $\sin\omega = 1/(1 + m_1/m)$. It is sufficient to impose only one boundary condition of the form (1) if the symmetry under permutations of identical particles is taken into account. As is well established, the regularization is required for the $L^P = 0^+$ states, therefore, namely this case will be considered here.

To analyze the proposed regularization, the problem is transformed to a system of the hyper-radial equations [5–7] and the solution near the triple-collision point, i.e., for small hyper-radius ρ ($\rho^2 = x^2 + y^2$) is studied. In the $\rho \to 0$ limit, the problem reduces to the two-body Schrödinger equation with the singular potential of the form $V_{\mathrm{sing}}(\rho) = \dfrac{\tilde{\gamma}^2 - 1/4}{\rho^2} + \dfrac{q}{\rho}$, which was multiply discussed in literature, e. g., in [7–9].

The results of [7] on the quantum problem for $V_{\mathrm{sing}}(\rho)$ are briefly summarized below. In the case $\tilde{\gamma}^2 \geq 1$ the problem is unambigouosly defined by the condition of square integrability. To define the problem for $0 \leq \tilde{\gamma}^2 < 1$ one should introduce an additional real-valued parameter b by imposing the boundary condition, e.g., of the form proposed in [7]

$$f(\rho) \xrightarrow[\rho\to 0]{} \rho^{\frac{1}{2}+\tilde{\gamma}} - \mathrm{sign}(b)|b|^{2\tilde{\gamma}}\rho^{\frac{1}{2}-\tilde{\gamma}}\left(1 + \frac{q\rho}{1-2\tilde{\gamma}}\right), \quad (2)$$

where the q-dependent term can be omitted for $0 \leq \tilde{\gamma}^2 < 1/4$. Finally, for $\tilde{\gamma}^2 < 0$, i.e., pure imaginary $\tilde{\gamma}$, one could define the unambiguous problem, e. g., by the requirement $f(\rho) \xrightarrow[\rho\to 0]{} \rho^{1/2}\sin(|\tilde{\gamma}|\log(\rho) + \delta)$ [8, 9]. This results in the asymptotic energy spectrum, $E_n \sim e^{-2\pi n/|\tilde{\gamma}|}$, which depends exponentially on the level's number. In fact, these considerations explain the Efimov effect in the three-body problem [1, 10].

Both $\tilde{\gamma}$ and q are single-valued functions of σ, which are obtained by solving the auxiliary eigenvalue problem on a hyper-sphere (for fixed ρ) with the boundary condition (1). Similar to [6, 7] one finds the equations

$$\sigma \sin\tilde{\gamma}\frac{\pi}{2} = \frac{\sin\tilde{\gamma}\omega}{\sin\omega} - \tilde{\gamma}\cos\omega\cos\tilde{\gamma}\frac{\pi}{2} \qquad (3)$$

for the two-component system and

$$\sigma \sin\tilde{\gamma}\frac{\pi}{2} = 4\sin\tilde{\gamma}\frac{\pi}{6} - \frac{\sqrt{3}}{2}\tilde{\gamma}\cos\tilde{\gamma}\frac{\pi}{2} \qquad (4)$$

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for three identical bosons. These equations implicitly determine monotonically increasing functions $\tilde{\gamma}^2(\sigma)$ shown in Fig. 1 both for the two-component system for $m/m_1 = 1$ and for three identical bosons.

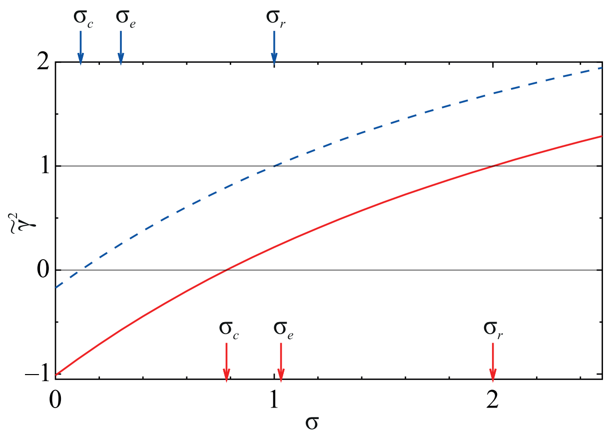

Fig. 1. (Color online) Dependence $\tilde{\gamma}^2$ on the regularization parameter σ are plotted by solid (red) line for three identical bosons and by dashed (blue) line for two identical bosons and a distinct particle of the same mass ($m/m_1 = 1$). Arrows indicate the critical values σ_c, σ_e, and σ_r at lower (upper) border for the former (latter) case

From the previous discussion one concludes that the Minlos–Faddeev regularization gives rise to four different types of description in four different intervals of the parameter σ separated by the critical values σ_c, σ_e, and σ_r, which correspond to $\tilde{\gamma}^2 = 0, 1/4$, and 1, respectively. The critical values for the two-component system follow from (3),

$$\sigma_c = \frac{2}{\pi}\left(\frac{\omega}{\sin\omega} - \cos\omega\right), \tag{5}$$

$$\sigma_e = \frac{1}{\sqrt{2}\cos\dfrac{\omega}{2}} - \frac{1}{2}\cos\omega, \tag{6}$$

and $\sigma_r = 1$. In the case of equal masses ($m/m_1 = 1$) Eqs. (5) and (6) give $\sigma_c = 2/3 - \sqrt{3}/\pi \approx 0.11534$ and $\sigma_e = 3\sqrt{3}/4 - 1 \approx 0.29904$. The same value of σ_c was given in [4]. From (4) one finds $\sigma_c = 4/3 - \sqrt{3}/\pi \approx$ ≈ 0.78200, $\sigma_e = 7\sqrt{3}/4 - 2 \approx 1.03109$, and $\sigma_r = 2$ for three identical bosons. The value of σ_c is the same as in [1–3].

Starting from the Minlos and Faddeev suggestion to modify the two-body zero-range interaction [1] it was declared [2–4] that the three-body problem becomes regularized, if the regularization parameter σ exceeds the critical value σ_c, i.e., if σ is sufficiently large to suppress the Efimov or Thomas effects.

In this work it is shown that the Minlos–Faddeev regularization gives different results in four intervals of the non-negative parameter σ, in particular, more strict condition on the regularization parameter, $\sigma > \sigma_r > \sigma_c$, is necessary for unambiguous description of the three-body problem. Within the interval $\sigma_c \le \sigma < \sigma_r$, it is necessary to set a boundary condition of the form (2) depending on a real-valued parameter b and the q-dependence can be safely omitted if $\sigma_c \le \sigma < \sigma_e$. At last, the Efimov or Thomas effect takes place for $\sigma < \sigma_c$ and the famous exponential asymptotic of the energy spectrum is obtained by imposing the boundary condition at $\rho \to 0$. To exemplify in details the main conclusions, three critical values σ_c, σ_e, and σ_r are determined both for the two-component system consisting of two identical bosons and a distinct particle and for the system consisting of three identical bosons. The effect of regularization is additionally demonstrated by the calculation of the bound-state energy for three identical bosons as a function of σ and b.

It is worthwhile to mention that the described scenario is anticipated for any problem, whose properties are essentially determined by the effective potential with the inverse square singularity, which strength goes through the critical values. Besides the two-component system consisting of two identical fermions and a distinct particle, which was described in [7], this scenario could be of importance also for the three-body problem in the mixed dimensions and in presence of the spin-orbit interaction.

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