THE $SU(3)$ BLACK HOLE

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The Einstein-Yang-Mills equations solution which is a $SU(3)$ black hole was built in this article.

In the Einstein's gravity theory there is the class of asymptotical flat gravitational fields, which are created by mass (the Schwarzschield's solution), mass and charge (the Reissner - Nordstrom's solution), rotated mass (the Kerr's solution). A similar solution was recently discovered, which describes the black holes with the gauge $SU(2)$ fields [1].

In this article the solution corresponding to the black holes with the gauge $SU(3)$ field is to be find.

Statement of the problem

The metric is choosen in the following spherically symmetric form:

$$ds^2 = \frac{\sigma^2 u}{r^2} dt^2 - \frac{r^2}{u} dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{2.1}$$

where the functions σ, u depend on r.

Introduce the Cartesian coordinates:

$$x^1 = r\cos\theta$$
$$x^2 = r\sin\theta\cos\phi$$
$$x^3 = r\sin\theta\sin\phi.$$

The Yang - Mills potential of the gauge group $SU(2)$ in this coordinates are sought in the form:

$$G_i^a = \frac{L^a_{ij} x^j}{r^2}(F(r)+1) - L^a_{jk}\frac{x^i x^j x^k}{r^4} \tag{2.2}$$

where $a = 1,2,...,8$; $i = 1,2,3$; the matrix L^a_{jk} expressed by means of matrix generators of Lie algebra $su(3)$ λ^a_{jk} :

$$L^1 = \lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad L^2 = -i\lambda^2 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$L^3 = \lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad L^4 = \lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$L^5 = -i\lambda^5 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad L^6 = \lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$L^7 = -i\lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \quad L^8 = \lambda^8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

The potential (2.2) in sphericall system of coordinates has the following form:

$$G_r^a = \Big\{ \sin 2\theta \cos\phi;\ \cos^2\theta - \sin^2\theta \cos^2\phi;\ \sin 2\theta \sin\phi;$$

$$0;\ \sin^2\theta \sin 2\phi;\ 0;\ \frac{1 - 3\sin^2\theta \sin^2\phi}{\sqrt{3}} \Big\} \frac{F(r)}{r} \tag{2.3'}$$

$$G_\theta^a = \Big\{ \cos 2\theta \cos\phi;\ \cos\phi;\ \frac{\sin 2\theta(\sin^2\phi - 2)}{2};\ \cos 2\theta \sin\phi;$$

$$\sin\phi;\ \frac{\sin 2\theta \sin^2\phi}{2};\ 0;\ -\frac{\sqrt{3}}{2} \sin 2\theta \sin 2\phi \Big\} (F(r) + 1) \tag{2.3''}$$

$$G_\phi^a = \Big\{ -\frac{\sin 2\theta \sin\phi}{2};\ -\frac{\sin 2\theta \sin\phi}{2};\ \frac{\sin^2\theta \sin 2\phi}{2};\ \frac{\sin 2\theta \cos\phi}{2};$$

$$\frac{\sin 2\theta \cos\phi}{2}\ \sin^2\theta\, cos 2\phi;\ \sin^2\theta;$$

$$-\frac{\sqrt{3}}{2} \sin^2\theta \sin 2\phi \Big\} (F(r) + 1). \tag{2.3'''}$$

Assume that the electrical field has the only nonzero component of Maxwell's tensor:

$$F_{tr} = -\frac{\partial A_t}{\partial r} = -A_t'. \tag{2.4}$$

The complete Lagrangian in that case can be written as:

$$L = \sqrt{-g} \left(-\frac{R}{16\pi\gamma} - \frac{1}{4e^2} W_{\mu\nu}^a W^{a\mu\nu} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right) \tag{2.5}$$

where R is a scalar curvature of metrics (2.1); $W_{\mu\nu}^a = \partial_\mu G_\nu^a - \partial_\nu G_\mu^a - f^{abc} G_\mu^b G_\nu^c$ is the gauge $SU(3)$ field strength; f^{abc} is the structural constant of the $SU(3)$ group; e is the gauge coupling constant; $\mu, \nu = 0, 1, 2, 3$. With a precision of unessential factor Lagrangian (2.5) can be written in following form:

$$L = \frac{\sigma' u}{r} + \sigma - \frac{\kappa\sigma}{r^4} \left[2u(rF' - F)^2 + 2uF^4 + \frac{r^2}{2}(F^2 - 1)^2 \right] + 8\pi\gamma \frac{r^2}{\sigma} (A_t')^2 \tag{2.6}$$

where $\kappa = 8\pi\gamma/e^2$. The variation (2.6) with A, σ, u and F leads to the following equation:

$$\left(\frac{r^2}{\sigma}A_t'\right)' = 0 \tag{2.7}$$

$$\left(\frac{u}{r}\right)' = 1 - \frac{\kappa}{r^4}\left[2u(rF' - F)^2 + 2uF^4 + \frac{r^2}{2}(F^2 - 1)^2\right] - \frac{q^2}{r^2} \tag{2.8}$$

$$\frac{\sigma'}{\sigma} = \frac{2\kappa}{r^3}[(rF' - F)^2 + F^4] \tag{2.9}$$

$$\left[\frac{u\sigma}{r^3}(rF' - F)\right]' = -\frac{\sigma u}{r^4}(rF' - F) + \frac{2\sigma u F^3}{r^4} + \frac{\sigma}{2r^2}F(F^2 - 1), \tag{2.10}$$

where q is a certain constant, which is proportional to electric charge.

The solution of Maxwell's equation (2.7) is:

$$A_t' = \frac{\sigma q}{r^2\sqrt{8\pi\gamma}}.$$

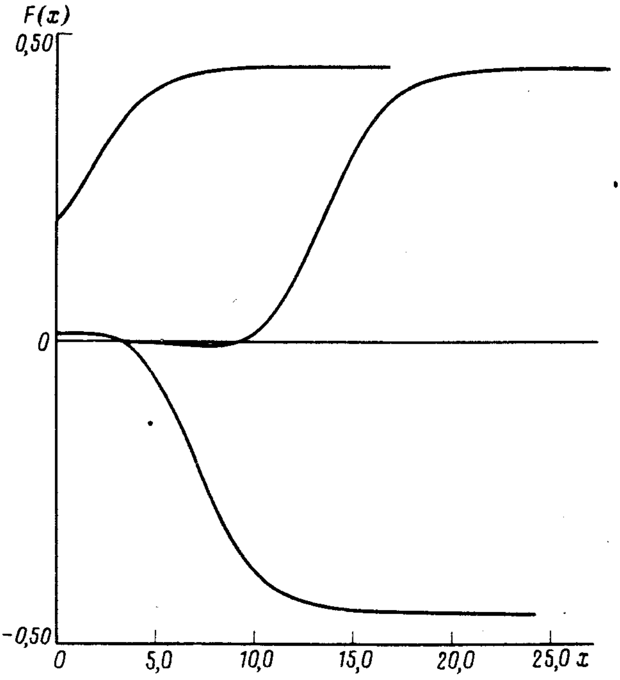

Fig.1. The function $V_0(x)$, $V_1(x)$ and $V_2(x)$. $\alpha = 1$, $Q = 0$

The function σ can be excluded from equation (2.8) and (2.10) with help of (2.9). Further assume that there is the event horizon at $r = r_H$, i.e. $u(r_H) = 0$. Here we introduce a dimensionless variables and the following magnitude:

$$e^x = \frac{r}{r_H},\ e^{2\phi} = \frac{u}{r_H^2},\ \alpha = \frac{\kappa}{r_H^2},\ Q = \frac{q}{r_H^2}.$$

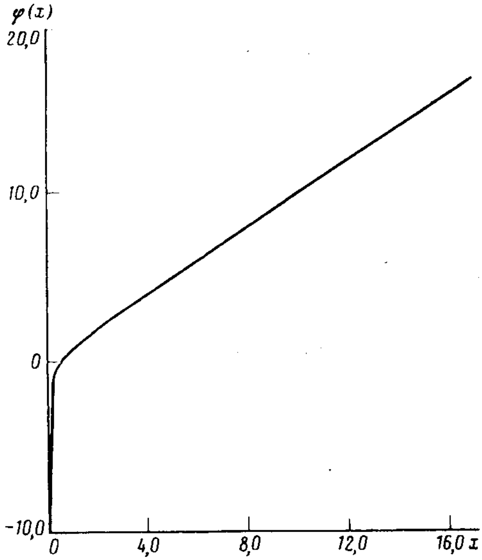

Fig.2. The function $\phi(x)$. $\alpha = 1$, $Q = 0$

After some simplification the Einstein - Yang - Mills equation can be written in the following form:

$$(F' - F)' + (F' - F)\left[e^{2x-2\phi} - \frac{\alpha(F^2-1)^2}{2}e^{-2\phi} - 1\right] = 2F^3 + \frac{F(F^2-1)}{2}e^{2x-2\phi} \tag{2.11}$$

$$\phi' + \frac{1}{2}\left[\frac{\alpha(F^2-1)^2}{2}e^{-2\phi} - e^{2x-2\phi}\right] = \frac{1}{2} - \alpha e^{-2x}[(F'-F)^2 + F^4] - Q^2 e^{-2\phi}. \tag{2.12}$$

here the ($'$) denotes the derivative of x.

The solution

Since the equation (2.11) is singular in point $x = 0$, a numerical solution of system (2.11)-(2.12) point $x = \Delta$ ($\Delta \ll 1$). If develop the function $\phi(x)$ and $F(x)$ as a series in powers of x, then one can see

$$[e^{2\phi(x=0)}]' = 1 - \frac{\alpha(F^{*2}-1)^2}{2} - Q^2 \tag{3.1}$$

$$F'(x=0) = F^* + \frac{F^*(F^{*2}-1)}{2\left[1 - \frac{\alpha(F^{*2}-1)^2}{2}\right]} \tag{3.2}$$

where $F^* = F(x=0)$.

Then:

$$\phi(x=\Delta) \approx \frac{1}{2}\ln\left\{\Delta\left[1-\frac{\alpha(F^{*2}-1)}{2}\right]\right\}$$

$$F(x=\Delta) \approx F^{*} + F'(x=0)$$

The numerical solution built on Runge - Kundt - Merson's method with the variable step. It has been established, that the solution of system (2.11)-(2.12) is analogical to the solution for $SU(2)$ field found in [1]. I.e. we have the regions of value F^{*} in which F^{*} approach either $+\infty$ or $-\infty$ by $x \to \infty$. This means that the points F^{*} on the boundary between this regions give the regular solution. As well as [1] this solution can be number of integer n, which shows how much is the solution $F(x)$ intersect the x-axis. Fig.1 presents the function $F_0(x)$, $F_1(x)$, $F_2(x)$ at $Q=0$. Here $F_0^{*}=0,192271...$; $F_1^{*}=0,00993494...$; $F_2^{*}=0,000428997....$ The asymptotically meaning for the solutions $F_n(x)$ is the value $F_\infty = \pm 1/\sqrt{5}$, that can be seen from (2.11). Fig.2 presents the one function $\phi(x)$ in that $\phi_0(x)$, $\phi_1(x)$ and $\phi_2(x)$ practicaly coincide. All the solutions $\phi_n(x) \to x$ by $x \to \infty$.

Thus the black holes filling chromodynamical $SU(3)$ field was built.

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1. M.S. Volkov, D.V. Gal'tsov, Yad. Fis. 51, 1171 (1990).