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CONFORMAL THEORY OF QUANTUM GRAVITY FROM INTEGRABLE HIERARCHIES

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We propose a direct derivation of a conformal field theory description of 2D quantum gravity + matter from the formalism of integrable hierarchies subjected to Virasoro constraints. The construction is based on a generalization of the Kontsevich parametrization of the Kadomtsev–Patiashvili (KP) times achieved by introducing Miwa parameters into it.

1. Introduction. The Matrix Models [1–3], besides their applications to matter+gravity systems [4–7], have also shown intriguing relations with integrable hierarchies subjected to Virasioro constraints [8–11] as well as with the intersection theory on the moduli space of curves [7,12,13]. However, a challenging problem remains of giving a direct proof of the equivalence between the 'hierarchical' formalism and the conformal field theory description of quantum gravity [14–16]. Another major task is to find the general solution to the Virasoro constraints on integrable hierarchies.

In this paper we show that these two problems are solved simultaneously, as the Virasoro constraints are in fact *solved* by a certain conformal theory. Our main tool will be a generalization of the Kontsevich parametrization [12] (see also [19,9]) of the KP times obtained by introducing into it parameters of the Miwa transformation [17,18] known from the KP hierarchy. It turns out that one has to allow the Miwa parameters to vary so as to be able to move between *different* (generalised) Kontsevich transformations: we will see that different Kontsevich transformations should be used depending on the operators one considers, which we call the Kontsevich–Miwa transform.

2. Virasoro action on the KP hierarchy. The KP hierarchy equations are imposed on the coefficients $w_n(x, t_1, t_2, t_3, \ldots)$ of a ψDiff operator [22] K of the form (with $D = \partial/\partial x$)

$$K = 1 + \sum_{n\geq 1} w_n D^{-n} \tag{1}$$

The wave function and the adjoint wave function are defined by

$$\psi(t,z) = K e^{\xi(t,z)}, \quad \psi^*(t,z) = K^{*-1} e^{-\xi(t,z)}, \quad \xi(t,z) = \sum_{r\geq 1} t_r z^r \tag{2}$$

where K^* is the formal adjoint of K. The wave functions are related to the tau function via

$$\psi(t,z) = e^{\xi(t,z)} \frac{\tau(t - [z^{-1}])}{\tau(t)}, \quad \psi^*(t,z) = e^{-\xi(t,z)} \frac{\tau(t + [z^{-1}])}{\tau(t)} \tag{3}$$

where $t \pm [z^{-1}] = (t_1 \pm z^{-1}, t_2 \pm \frac{1}{2} z^{-2}, t_3 \pm \frac{1}{3} z^{-3}, \ldots)$.

The Virasoro action on the tau function is implemented by the generators,

$$\begin{aligned} \widehat{L}_{p>0} &= \tfrac{1}{2} \textstyle\sum_{k=1}^{p-1} \frac{\partial^2}{\partial t_{p-k} \partial t_k} + \sum_{k\geq 1} k t_k \frac{\partial}{\partial t_{p+k}} + (a_0 + (J - \frac{1}{2})p) \frac{\partial}{\partial t_p} \\ \widehat{L}_0 &= \textstyle\sum_{k\geq 1} k t_k \frac{\partial}{\partial t_k} + \frac{1}{2} a_0^2 - \frac{1}{2} \left(J - \frac{1}{2}\right)^2 \\ \widehat{L}_{p<0} &= \textstyle\sum_{k\geq 1} (k-p) t_{k-p} \frac{\partial}{\partial t_k} + \frac{1}{2} \sum_{k=1}^{-p-1} k(-p-k) t_k t_{-p-k} + \\ &+ (a_0 + (J - \tfrac{1}{2})p)(-p) t_{-p} \end{aligned} \tag{4}$$

which satisfy the Virasoro algebra with central charge $-12(J^2 - J + \frac{1}{6})$.

3. Miwa–Kontsevich transform. The Miwa reparametrization of the KP times is accomplished by the substitution

$$t_r = \frac{1}{r} \sum_j n_j z_j^{-r} \tag{5}$$

where $\{z_j\}$ is a set of points on the complex plane. By the Kontsevich transform we understand the dependence, via eq.(5), of t_r on the z_j for *fixed* n_j. To recast the Virasoro constraints $\widehat{L}_{n\geq -1}\tau = 0$ into the Kontsevich parametrization, note that picking out the involved $\widehat{L}$'s amounts to retaining in the energy-momentum tensor $\widehat{T}(z) = \sum_{p\in\mathbb{Z}} z^{-p-2} \widehat{L}_p$ only terms with z to negative powers:

$$\widehat{T}^{(\infty)}(v) = \sum_{n\geq 0} v^{-n-1} \frac{1}{2\pi i} \oint dz z^n \widehat{T}(z) = \frac{1}{2\pi i} \oint dz \frac{1}{v - z} \widehat{T}(z) \tag{6}$$

where v is from a neighbourhood of the infinity and the integration contour encompasses this neighbourhood.

A crucial simplification is achieved by evaluating $\widehat{T}^{(\infty)}(v)$ only at a point from the above set $\{z_j\}$ (one has to take care that they lie in the chosen neighbourhood). We need to express the Virasoro action on the tau function through the $\partial/\partial z_j$ derivatives, but the equation relating t_r and z_j does not allow us to substitute $\partial/\partial t_r$ in terms of $\partial/\partial z_j$. It is only after we evaluate the residues in (6) that we find the t-derivatives to arrange into the combinations which are just the desired $\partial/\partial z_j$'s, apart from the term $-\left(J - \frac{1}{2} - \frac{1}{2n_i}\right) \sum_{r\geq 1} r z_i^{-r-2} \partial/\partial t_r$ which should thus be set to zero by choosing

$$n_i = \frac{1}{2J-1} \equiv \frac{1}{Q} \tag{7}$$

In this way we arrive at the action of $\widehat{T}^{\infty}(z_i)$ on the tau function in the Kontsevich parametrization given by the operator

$$\mathcal{T}_{\{n\}}(z_i) = -\frac{Q^2}{2}\frac{\partial^2}{\partial z_i^2} - \sum_{j\neq i} \frac{1}{z_j - z_i}\left(\frac{\partial}{\partial z_j} - Q n_j \frac{\partial}{\partial z_i}\right) \tag{8}$$

This operator depends on the collection of the n_j with $j \neq i$. In particular, if one wishes *all* the $\widehat{T}^{(\infty)}(z_j)$ to carry over to the Kontsevich variables along with $\widehat{T}^{(\infty)}(z_i)$, all the n_j have to be fixed to the same value (7). Then, one gets "symmetric" operators

$$\mathcal{T}(z_i) = -\frac{Q^2}{2}\frac{\partial^2}{\partial z_i^2} - \sum_{j\neq i} \frac{1}{z_j - z_i}\left(\frac{\partial}{\partial z_j} - \frac{\partial}{\partial z_i}\right) \tag{9}$$

These satisfy the centreless algebra spanned by the $\{\widehat{L}_{n\geq -1}\}$ Virasoro generators. Then, if one starts with the *Virasoro-constrained* KP hierarchy, *i.e.*, $\widehat{T}^{(\infty)}(z)\tau = 0$, one ends up with the KP Virasoro *master equation* (cf. ref.[20]) $\mathcal{T}(z_i).\tau\{z_j\} = 0$.

5. Conformal field theory from Virasoro constraints. Now, introduce a conformal theory of a $U(1)$ current $j(z) = \sum_{n\in\mathbf{Z}} j_n z^{-n-1}$ and an energy-momentum tensor $T(z) = \sum_{n\in\mathbf{Z}} L_n z^{-n-2}$:

$$\begin{aligned} [j_m, \quad j_n] &= km\delta_{m+n,0} \\ [L_m, \quad L_n] &= (m-n)L_{m+n} + \tfrac{d+1}{12}(m^3 - m)\delta_{m+n,0} \\ [L_m, \quad j_n] &= -nj_{m+n} \end{aligned} \tag{10}$$

(We have parametrized the central charge as $d+1$). Let Ψ be a primary field with conformal dimension Δ and $U(1)$ charge q. Then, in the standard setting of [21], we find that the level-2 vector

$$|\Upsilon\rangle = \left(\alpha L_{-1}^2 + L_{-2} + \beta j_{-2} + \gamma j_{-1} L_{-1}\right)|\Psi\rangle \tag{11}$$

is primary provided

$$\alpha = \frac{k}{2q^2}, \quad \beta = -\frac{q}{k} - \frac{1}{2q}, \quad \gamma = -\frac{1}{q}, \quad \Delta = -\frac{q^2}{k} - \frac{1}{2} \tag{12}$$

with q and Δ given by,

$$\frac{q^2}{k} = \frac{d - 13 \pm \sqrt{(25-d)(1-d)}}{24}, \quad \Delta = \frac{1 - d \mp \sqrt{(25-d)(1-d)}}{24}. \tag{13}$$

Factoring out the state (11) leads in the usual manner to the equation

$$\left\{\frac{k}{2q^2}\frac{\partial^2}{\partial z^2}-\frac{1}{q}\sum_j\frac{1}{z_j-z}\left(q\frac{\partial}{\partial z_j}-q_j\frac{\partial}{\partial z}\right)+\right.$$

$$\left.+\frac{1}{q}\sum_j\frac{q\Delta_j-q_j\Delta}{(z_j-z)^2}\right\}\langle\Psi(z)\Psi_1(z_1)\ldots\Psi_n(z_n)\rangle=0 \tag{14}$$

where Ψ_j are primaries of dimension Δ_j and $U(1)$ charge q_j. In particular,

$$\left\{\frac{k}{2q^2}\frac{\partial^2}{\partial z_i^2}+\sum_{j\neq i}\frac{1}{z_i-z_j}\left(\frac{\partial}{\partial z_j}-\frac{\partial}{\partial z_i}\right)\right\}\langle\Psi(z_1)\ldots\Psi(z_n)\rangle=0 \tag{15}$$

Writing the Hilbert space as (matter) $\otimes$ (current) $\equiv \mathcal{M}\otimes\mathcal{C}$, $|\Psi\rangle=|\psi\rangle\otimes|\tilde{\Psi}\rangle$, we introduce the matter Virasoro generators l_n by,

$$L_n=l_n+\hat{L}_n\equiv l_n+\frac{1}{2k}\sum_{m\in\mathbf{Z}}:j_{n-m}j_m: \tag{16}$$

They then have central charge d. It turns out that

$$|\Upsilon\rangle=\left(\frac{k}{2q^2}l_{-1}^2+l_{-2}\right)|\Psi\rangle \tag{17}$$

and thus we are left with a null vector in the matter Hilbert space $\mathcal{M}$. Now, the dimension of $|\psi\rangle$ in the matter sector,

$$\delta=\Delta-\frac{1}{2k}q^2=\frac{5-d\mp\sqrt{(25-d)(1-d)}}{16}, \tag{18}$$

is of course (for the appropriate values of d) that of the '21' operator of the minimal model with central charge d [24,25].

The above can now be used to solve the Virasoro constraints on the KP hierarchy by assuming the ansatz [23]

$$\tau\{z_j\}=\lim_{n\to\infty}\langle\Psi(z_1)\ldots\Psi(z_n)\rangle \tag{19}$$

Then, comparing eqs.(14) and (8), one finds

$$Q^2=-\frac{k}{q^2}=\frac{13-d\pm\sqrt{(25-d)(1-d)}}{6}, \tag{20}$$

and d is therefore determined in terms of the parameter Q from (4) (where $J=\frac{Q+1}{2}$), as $d=13-3Q^2-\frac{12}{Q^2}$.

To see what the matter theory field operators are which can be derived from the Virasoro constraints, consider the form the $\hat{L}_{n\geq-1}$-constraints take for the wave function of the hierarchy, $w(t,z_k)\equiv e^{-\xi(t,z_k)}\psi(t,z_k)$, which should now become a function of the z_j, $w\{z_j\}(z_k)$. More precisely, consider the 'unnormalized' wave function $\bar{w}\{z_j\}(z_k)=\tau\{z_j\}w\{z_j\}(z_k)$. Then the use of the Kontsevich transform at the Miwa point $n_j = 1/Q$, $j \neq k$ and $n_k = -1$, gives[1)]

$$\bar{w}\{z_j\}(z_k) = \left\langle \prod_{j\neq k} \Psi(z_j) \cdot \Xi(z_k) \right\rangle \tag{21}$$

where Ξ is a primary field with the $U(1)$ charge qQ and dimension $Q\Delta$. Now we choose in (20) the branch of the square root $\sqrt{Q^2}$ so that Q be positive for $d < 1$:

$$Q = \frac{1}{2}\sqrt{\frac{25-d}{3}} \pm \frac{1}{2}\sqrt{\frac{1-d}{3}} \equiv -\frac{-Q_{\rm L} \mp Q_{\rm m}}{2} \equiv -\alpha_{\mp} \tag{22}$$

(with the upper/lower signs corresponding to those in (20)). This establishes the physical meaning of the background charge Q present initially in the Virasoro constraints. (Note that it has entered explicitly in the Kontsevich transform through (7).) Now, the dimension of Ξ is equal to $\mp\frac{1}{2}Q_{\rm m}$, which implies in turn that its dimension in the matter sector equals

$$\mp\frac{1}{2}Q_{\rm m} - \frac{1}{2k}(qQ)^2 = \mp\frac{1}{2}Q_{\rm m} + \frac{1}{2} \equiv \begin{cases} 1 - J_{\rm m} \\ J_{\rm m} \end{cases} \tag{23}$$

where $J_{\rm m}$ is the conformal 'spin' (dimension) of a bc system. Thus, the wave function is associated (for, say, the lower signs) with the b-field B of a bc system. The adjoint wave function is then similarly related to the corresponding c field C: for instance, the function $\tau(t - [z_k^{-1}] + [z_l^{-1}])$ is proportional to the correlation function[2)]

$$\left\langle \prod_{\substack{j\neq k\\ j\neq l}} \Psi(z_j) \exp\left(\frac{qQ}{k}\int^{z_k} j\right) B(z_k) \exp\left(-\frac{qQ}{k}\int^{z_l} j\right) C(z_l) \right\rangle =$$

$$= \left\langle \prod_{\substack{j\neq k\\ j\neq l}} \Psi(z_j)(z_k - z_l) \exp\left(\frac{qQ}{k}\int_{z_l}^{z_k} j\right) B(z_k)C(z_l) \right\rangle \tag{24}$$

Note that this is ***not*** the system of free fermions underlying the construction of general (i.e., not Virasoro-constrained) tau functions. By bosonization one gets a scalar φ with the energy-momentum tensor $T_{\rm m} = -\frac{1}{2}\partial\varphi\partial\varphi + \frac{i}{2}Q_{\rm m}\partial^2\varphi$, thus establishing the relation with minimal models [21,24,25] (for appropriate values of the central charge $d = 1 - 3Q_{\rm m}^2$).

Further, as to the theory in C, recall that we have $[j_m, j_n] = km \times \delta_{m+n,0}$, $j_{n>0}|\Psi\rangle = 0$, $j_0|\Psi\rangle = q|\Psi\rangle$ with negative q^2/k (for $d < 1$). To

[1)] To obtain the insertion into the correlation function (21) at the point z_k of the operator we are interested in by itself, rather than its fusion with the 'background' Ψ, we use the Kontsevich transform at the value of the Miwa parameter $n_k = -1$ instead of $\frac{1}{Q} - 1$. This means that we are in fact considering $\bar{w}\{z_j\}_{j\neq k}(z_k)$. Similar remarks apply to other correlation functions considered below.

[2)] Therefore the whole 'Borel' subalgebra of the $W_\infty(J)$ algebra [26], which is the symetry algebra of the Virasoro-constrained KP hierarchy [27], is represented in terms of the bilocal operator insertions, read off from (24), placed at points from the *fixed set* $\{z_j\} \times \{z_{j'}\} \subset \mathbf{CP}^1 \times \mathbf{CP}^1$.

see what the current corresponds to in the KP theory, consider the correlation function with an extra insertion of an operator which depends on only j:

$$\left\langle \prod_{j\neq k,\ j\neq l} \Psi(z_j) \exp \frac{2\Delta}{q} \int_{z_l}^{z_k} j \right\rangle \tag{25}$$

The decoupling equation states that the correlation function (25) coincides, up to a constant, with the tau function $\tau(t)$ evaluated at the Miwa point $n_j = \frac{1}{Q}$, $j \neq k$, $j \neq l$, $n_k = Q - (2/Q)$, $n_l = (2/Q) - Q$. Here, $|Q - (2/Q)| = Q_{\rm m}$; abusing the notations, the function we are considering can be written as $\tau\{z_j\}_{\substack{j\neq k\\ j\neq l}}(t - Q_{\rm m}[z_k^{-1}] + Q_{\rm m}[z_l^{-1}])$.

The balance of dimensions and $U(1)$ charges of both the Ψ and Ξ operators follows a particular pattern: we find from (14) that $\Delta_j = \Delta \frac{q_j}{q}$. Then, the dimension in the matter sector M is equal to

$$\delta_j = \Delta_j - \frac{q_j^2}{2k} = \Delta \frac{q_j}{q} - \frac{q_j^2}{2k} \tag{26}$$

As the coefficient at the term linear in $q_j/\sqrt{-k}$ is $\frac{1}{2}Q_{\rm m}$, this equation will always be satisfied for the matter operators $e^{i\gamma\varphi}$ provided $q_j/\sqrt{-k} = \gamma$! – Thus the 'dressing' prescription inherited from the KP hierarchy says that the coefficients with which the two scalars φ and ϕ enter the exponents coincide (to be precise, up to the factor of i). Therefore, although the field content is the same as in ref.[15], the David-Distler-Kawai formalism is not recovered directly from the KP hierarchy.

The 'bulk' dimensions Δ_j, rather than being equal to 1, are related to the gravitational scaling dimensions of fields. Indeed, evaluating the gravitational scaling dimension of ψ according to [14–16],

$$\hat{\delta}_\pm = \frac{\pm\sqrt{1-d+24\delta} - \sqrt{1-d}}{\sqrt{25-d} - \sqrt{1-d}} \tag{27}$$

one would find

$$\hat{\delta}_+ = \frac{3}{8} \pm \frac{d - 4 - \sqrt{(1-d)(25-d)}}{24} \tag{28}$$

with the sign on the RHS corresponding to that in (13) and the subsequent formulae. In particular, choosing the *lower* signs throughout, we have $\hat{\delta}_+ = \Delta + \frac{1}{2}$. More generally, the gravitational scaling dimensions corresponding to (26) equal

$$\hat{\delta}_{j+} = -\frac{q_j q}{k} = \Delta_j + \frac{1}{2}\frac{q_j}{q} = \Delta_j + \frac{1}{2} Q \frac{q_j}{\sqrt{-k}} \tag{29}$$

(Again, this is valid for the '+'-gravitational scaling dimensions *and* the lower signs in eqs.(20) etc., i.e., for only one out of four possibilities to choose the signs.)

5. Concluding remarks. 1. Various aspects of the conversion of Virasoro constraints into decoupling equations deserve more study from the 'Liouville' point of view. The Kontsevich-type matrix integral whose Ward identities coincide with our master equation may thus provide a candidate for a discretized definition of the Liouville theory.

2. For the matter central charge d from the minimal-models series, how can the *higher* null-vector decoupling equations be arrived at starting from the Virasoro-constrained hierarchies?

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