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QUANTUM HALL AND CHIRAL EDGE STATES IN THIN ^{3}He-A FILM

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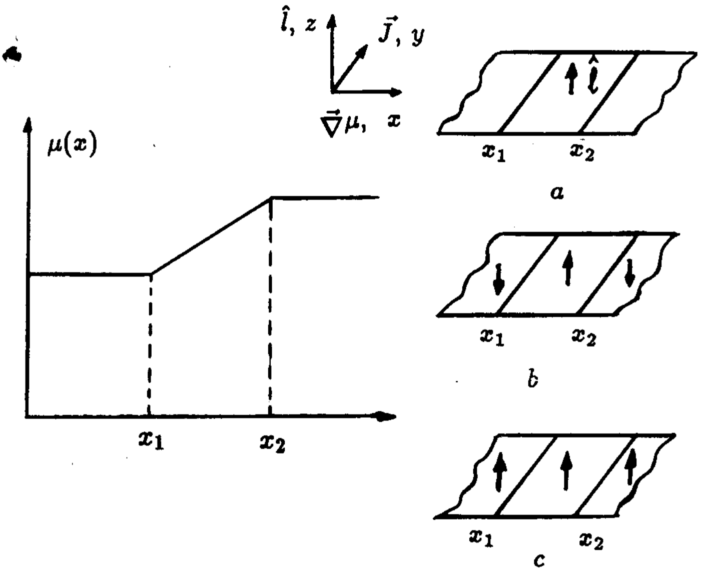
The chiral gapless excitations on the boundary of the superfluid ^{3}He-A film give rise to quantization of the transverse (Hall) conductivity in the absence of magnetic field.

The edge states of fermions have been recently discussed for a droplet of two-dimensional electron gas exhibiting the Quantum Hall Effect (QHE) under applied magnetic field. [1-3] The fermions on the boundary of the droplet are chiral and gapless and thus represent the only low-energy fermionic excitations in this system, since there is a finite energy gap for fermions within the droplet. The neutral superfluid ^{3}He-A film represents the QHE without magnetic field[4]: the response of the particle current on the gradient of the chemical potential applied in the transverse direction (analog of Hall conductivity) exhibits quantization. However as was shown in Ref.[4] this quantization rule is only approximate and is valid in the extreme limit of the small gap as compared with the Fermi energy. Here we show that in the geometry which leads to the existence of the chiral edge excitations in the ^{3}He-A film the quantization of the Hall conductivity is exact.



Profile of the chemical potential applied to the strip of the ^{3}He-A film in three different geometries: (a) there is no $\hat{l}$ vector outside the layer; (b) $\hat{l}$ is oriented in the opposite direction outside the layer; (c) $\hat{l}$ is oriented everywhere in the same direction.

For the neutral ^{3}He-A film the role of the magnetic field is played by the spontaneous orbital angular momentum of the Cooper pairs, which violates the time inversion and 2d space inversion symmetries. The direction of the momentum, denoted by unit vector $\hat{l}$, is fixed along the normal to the film: $\hat{l} = \pm\hat{z}$. We consider here the QHE in the following geometry (see Figure): the difference of chemical potentials $\mu(x_2) - \mu(x_1)$ is applied to the strip $x_1 < x < x_2$ of the film with given orientation of $\hat{l} = \hat{z}$ and the mass current J is measured in the direction y. We consider three different geometries: (1) outside the layer there is no ^{3}He-A, instead for example there is the planar state which has no orbital momentum (Fig.a) ; (2) the ^{3}He-A film with an opposite orientation of $\hat{l} = -\hat{z}$ (Fig.b) is outside the strip; (3) the ^{3}He-A film everywhere has the same orientation of $\hat{l}$ (Fig.3). Our results for the total current in the strip for these cases are:

$$J^{(1)} = \frac{N}{2}\frac{m_3}{2\pi\hbar}(\mu(x_2) - \mu(x_1)) \quad , \tag{1}$$

$$J^{(2)} = N\frac{m_3}{2\pi\hbar}(\mu(x_2) - \mu(x_1)) - \frac{\hbar}{4}(\rho(x_2) - \rho(x_1)) \quad , \tag{2}$$

$$J^{(3)} = \frac{\hbar}{4}(\rho(x_2) - \rho(x_1)) \quad , \tag{3}$$

where ρ is the particle density per unit area of the film, m_3 is the mass of ^{3}He atom, and N is related to the number of the chiral fermions on the boundary of the layer, which depends on the film thickness and increases with increase of the thickness. N is even for the ^{3}He-A film and is any integer for the ^{3}He-A$_1$ film. Since this result does not depend on the details of the system we calculate the current using the simplest model for the ^{3}He-A film.

In the thin ^{3}He film the dimensional quantization along z becomes important, and the quasiparticle spectrum in the normal ^{3}He film depends on the index q of discrete level of the motion along z and on two-dimensional momentum $\vec{k} = (k_x, k_y)$. For the simplest model of the noninteracting levels q the spectrum of the particles on each level of a normal film is

$$\varepsilon_q(\vec{k}) = \varepsilon_q(0) + \frac{k^2}{2m_3} \quad , \tag{4}$$

where $\varepsilon_q(0) \sim \hbar^2 q^2/m_3 a^2$ with a being the thickness of the film. On each level q, which is below the chemical potential, $\varepsilon_q(0) < \mu$, the Fermi-liquid is formed with its own Fermi-momentum k_{Fq}

$$\frac{k_{Fq}^2}{2m_3} = \mu - \varepsilon_q(0) \quad . \tag{5}$$

The Cooper pairing leads to nondiagonal matrix elements between particle and hole states. The relevant Bogoliubov-Nambu matrix for the fermions on the level q in the ^{3}He-A film is 2×2 matrix, if one discards the spin part of the order parameter and neglects the interlevel interaction. It is expressed in terms of two vectors $\hat{e}^1$ and $\hat{e}^2$ in the (x, y) plane:

$$\mathrm{H}_q = (\varepsilon_q(\vec{k}) - \mu)\tau_3 + c_q\vec{k}\cdot\hat{e}^1 \;\tau_1 - c_q\vec{k}\cdot\hat{e}^2 \;\tau_2 \quad , \tag{6}$$

where $\vec{\tau}$ are the Pauli matrices in the particle-hole space, and c_q are the amplitudes of the nondiagonal elements. In the equilibrium ^{3}He-A film $\hat{e}^1 \perp \hat{e}^2$, $|\hat{e}^1| = |\hat{e}^2| = 1$ and $\hat{l} = \hat{e}^1 \times \hat{e}^2$. The energy spectrum

$$E_q^2(\vec{k}) = (\varepsilon_q(\vec{k}) - \mu)^2 + c_q^2[(\vec{k}\cdot\hat{e}^1)^2 + (\vec{k}\cdot\hat{e}^2)^2] \tag{7}$$

is nowhere zero in the equilibrium ^{3}He-A film, like in the two-dimensional electron gas in the regime of the QHE.

The zeroes in the spectrum appear at $x = x_1$ and at $x = x_2$, i.e. at the edges of the ^{3}He-A strip. We consider first the case (2), when the edges of the layer are borders between domains with different $\hat{l}$ orientation. The topological properties of the spectrum are insensitive to the details, so we choose the simplest realization[5] for $\hat{e}^1(x)$ and $\hat{e}^2(x)$:

$$\hat{e}^1(x) = \hat{x} \quad , \quad \hat{e}^2(x) = -\hat{y}\ \tanh(\frac{x - x_1}{\xi_B})\tanh(\frac{x - x_2}{\xi_B}) \quad , \tag{8}$$

where ξ_B is the size of the domain boundary, which is of order coherence length, $\xi_B \gg k_F^{-1}$, we also assume that $\xi_B \ll (x_2 - x_1)$. Far from the boundaries $\hat{e}^2 = -\hat{y}$ at $x < x_1$ and $x > x_2$, and $\hat{e}^2 = \hat{y}$ at $x_1 < x < x_2$, which corresponds to $\hat{l} = -\hat{z}$ at $x < x_1$ and $x > x_2$, and $\hat{l} = \hat{z}$ at $x_1 < x < x_2$.

Since $\xi_B \gg k_F^{-1}$ we may first consider the spectrum in the semiclassical approximation, in which the spectrum depends both on the momentum and the coordinate: $E_q(\vec{k}, \vec{r})$ in Eq.(7) with $\hat{e}^1(x)$ and $\hat{e}^2(x)$ from Eq.(8). The energy becomes zero at the lines ($x = x_1$, $k_x = 0$, $k_y = \pm k_{Fq}$) and ($x = x_2$, $k_x = 0$, $k_y = \pm k_{Fq}$) in the 4-dimensional $(\vec{k}, \vec{r})$- space. These are the straight lines along the y axis of $4d$ space. These manifolds of zeroes have the topological stability. The explicit expression for the topological invariant, which supports the stability of the zeroes, may be constructed[4] in terms of the Green's function matrix $\mathrm{G}_{qp}(\omega, \vec{k}, x)$:

$$m = \frac{1}{24\pi^2} e_{\mu\nu\lambda\gamma}\ \mathrm{tr} \int dS^\gamma\ \ \mathrm{G}\partial_\mu \mathrm{G}^{-1} \mathrm{G}\partial_\nu \mathrm{G}^{-1} \mathrm{G}\partial_\lambda \mathrm{G}^{-1} \quad . \tag{9}$$

For given value of y the integral is taken over the 3d sphere in 4d space (ω, k_x, k_y, x) about each zero point of the spectrum, say, ($\omega = 0$, $x = x_1$, $k_x = 0$, $k_y = k_{Fq}$). This integral is $m = 1$ for zeroes at $x = x_1$ and $m = -1$ for zeroes at $x = x_2$, which can be checked in the model of the noninteracting levels, where the Green function matrix is diagonal in level q indices:

$$\mathrm{G}_{qp}(\omega, \vec{k}, x) = \delta_{qp} \frac{1}{i\omega + \mathrm{H}_q(\vec{k}, x)} \quad . \tag{10}$$

The number of zeroes in the semiclassical energy spectrum $E_q(\vec{k}, \vec{r})$ is thus $4q_0$ for each domain boundary, where $2q_0$ is the number of the Fermi-liquids in normal state: q_0 is the number of the levels of the quantized motion along z below μ and we take into account the double degeneracy over spin; for the ^{3}He-A$_1$, where only one spin component forms the Cooper pairs, there is no factor 2.

The index theorem relates the number of zeroes in the semiclassical spectrum with the same number $4q_0$ of the gapless fermionic modes localized on the boundary in the exact quantum-mechanical problem. The exact energy spectrum of the fermions, $E_n(k_y)$, depends on the momentum k_y along the domain boundary. In the simplest realization of the structure of the domain boundary,[5] the Hamiltonian which defines the spectrum, say, at $x = x_1$ and $k_y \approx k_{Fq}$ is

$$\mathrm{H}_q = v_{Fq}(k_y - k_{Fq})\tau_3 + c_q\tau_1(-i\frac{\partial}{\partial x}) - c_q k_{Fq} \tanh\frac{x - x_1}{\xi_B}\ \tau_2 \quad . \tag{11}$$

Each Hamiltonian has zero-mode eigenfunction, the spinor $\Psi = (u(x), v(x))$:

$$\Psi = (\ 0\ ,\ \mathrm{ch}^{-s}\frac{x}{\xi_B},) \quad , \quad s = k_{Fq}\xi_B \quad . \tag{12}$$

Each mode produces the gapless branch of the fermionic spectrum, which crosses zero value at $k_y = k_{Fq}$. These are one-dimensional Fermi-liquids.

It is important that the symmetry with respect to $k_y \to -k_y$ is broken here: in the vicinity of the Fermi-points $\pm k_{Fq}$ the spectrum is

$$E_0(q,k_y) = \mathrm{sign}(k_y)\ \ (\epsilon_q(k_y)-\mu) \approx v_{Fq}(k_y \mp k_{Fq}) \quad , \tag{13}$$

which corresponds to the right moving zero fermionic modes on the domain boundary. Altogether there are $4q_0$ right moving gapless fermions localized on the boundary at $x = x_1$ and the same amount of the left moving chiral gapless fermions at $x = x_2$.

Due to this asymmetry there is a net linear momentum and therefore the ground-state mass current in each of the domain boundaries. The magnitude of the vacuum current may be obtained using the gradient expansion,[6] which holds since $\xi_B k_F \gg 1$. The expression for the current in the inhomogeneous order parameter field may be obtained in terms of the phase of the order parameter[7]:

$$\vec{j}(\vec{r}) = \frac{1}{2}\sum_{q,\vec{k}} \vec{k} n_q(\vec{k},\vec{r})\frac{\partial}{\partial \vec{r}}\Phi(\vec{k},\vec{r}) + \frac{1}{2}\frac{\partial}{\partial r_i}[\ \sum_{q,\vec{k}} \vec{k} n_q(\vec{k},\vec{r})\frac{\partial}{\partial k_i}\Phi(\vec{k},\vec{r})\ ]-$$

$$-\frac{1}{2}\sum_{q,\vec{k}} \vec{k} n_q(\vec{k},\vec{r})\ \left(\frac{\partial}{\partial \vec{r}}\cdot\frac{\partial}{\partial \vec{k}} - \frac{\partial}{\partial \vec{k}}\cdot\frac{\partial}{\partial \vec{r}}\right)\ \Phi(\vec{k},\vec{r}) \quad , \tag{14}$$

$$n_q(\vec{k}) = \frac{1}{2}\left(1-\frac{\epsilon_q(\vec{k})-\mu}{E_q(\vec{k},\vec{r})}\right) \quad , \quad \tan\ \Phi(\vec{k},\vec{r}) = \frac{\vec{k}\cdot\hat{e}^2(x)}{\vec{k}\cdot\hat{e}^1(x)} \ . \tag{15}$$

The first term in Eq.(14) has no contribution to the current along y. The integration of the second term, which is a full derivative, over x from $-\infty$ to $+\infty$ leads to the regular contribution

$$\vec{J}_{\mathrm{regular}} = -\frac{\hbar}{4}\hat{y}\ \ (\rho(x_2)-\rho(x_1)), \tag{16}$$

which exists even in the absence of the edge chiral states and is related to the edge currents produced by the orbital momentum of Cooper pairs $\vec{L} = \frac{1}{2}\hbar\rho\hat{l}$ (it is $\hbar\hat{l}$ per each two ^{3}He atoms of this p-wave superfluid):

$$\vec{J}_{\mathrm{regular}} = \frac{1}{2}\int_{-\infty}^{+\infty} dx\ \ \vec{\nabla}\times\vec{L}. \tag{17}$$

The third term is concentrated in the domain boundaries and gives the contribution from the chiral edge states. Since

$$n_q(\vec{k},\vec{r})\ \left(\frac{\partial}{\partial \vec{r}}\cdot\frac{\partial}{\partial \vec{k}} - \frac{\partial}{\partial \vec{k}}\cdot\frac{\partial}{\partial \vec{r}}\right)\Phi(\vec{k},\vec{r}) = \mathrm{sign}(k_y)\ \ 2\pi[\delta(x-x_1)-\delta(x-x_2)]\delta(k_x)\Theta(k_{qF}-k) \ , \tag{18}$$

one obtains for this anomalous contribution to the current

$$\vec{J}_{\mathrm{an}}(x_2)+\vec{J}_{\mathrm{an}}(x_1) = \hat{y}\ \frac{1}{2\pi\hbar}\ \sum_q(\mu(x_2)-\epsilon_q(0))\Theta(\mu(x_2)-\epsilon_q(0))-$$

$$-\hat{y}\ \frac{1}{2\pi\hbar}\sum_q(\mu(x_1)-\epsilon_q(0))\Theta(\mu(x_1)-\epsilon_q(0)) \ = \hat{y}\frac{N}{2\pi\hbar}(\mu(x_2)-\mu(x_1)) \quad , \tag{19}$$

where $N = 2q_0$ ($N = q_0$ for the ^{3}He-A$_1$ film).

The anomalous current may be also obtained directly from the exact spectrum of the chiral mode in the vicinity of the Fermi-point in Eq.(13). The change $\delta\mu$ in the chemical potential leads to flow of the fermionic levels k_y through the

Fermi-points and therefore the linear momentum $\propto \delta\mu$ is created from the vacuum. The response of the anomalous current $\hat{y}\sum_{q,k_y} k_y \Theta(-E_0(q,k_y))$ to $\delta\mu$ is thus

$$\frac{dJ_{\rm an}}{d\mu} = \frac{m_3}{2\pi\hbar}\sum_q \Theta(\mu - \varepsilon_q(0)) = N\frac{m_3}{2\pi\hbar} \quad , \qquad (20)$$

which is the variation of Eq.(19). This effect of the momentum creation is the manifestation of the same chiral anomaly, which was discussed for the bulk ^{3}He-A [8,9].

The total current in the geometry of Fig.b is the sum of the regular and anomalous terms, Eq.(16) + Eq.(19), which leads to Eq.(2). In the geometry of Fig.a there is no $\hat{l}$ outside the layer, and the regular contribution from Eq.(17) is absent. As for the anomalous contribution one should retain only half of it since the edge of the ^{3}He-A contains twice less the number of the chiral fermions; this leads to Eq.(1). In the geometry of Fig.c there are no edge states at all and one has only the regular contribution in Eq.(3); this case considered in Ref.[4] has no exact quantization, though in the limit of weak interaction between the fermions the quantity $\partial\rho/\partial\mu$ approaches the step-wise behavior. The exact quantization of the current is related only to the chiral edge states. From Figure and Eqs.(1-3) it also follows that the current obeys the summation rule, which follows from that for the Cooper pair orbital momentum: in particular $\vec{J}^{(2)} + \vec{J}^{(3)} = 2\vec{J}^{(1)}$.

Note that the response of the current in this analog of QHE is quantized in terms of the same topological number N as the Chern-Simons term in the ^{3}He-A film which determines the spin and quantum statistics of the particle-like solitons.[10]

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