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Review on special geometry and mirror symmetry for Calabi–Yau manifolds (Mini-review)

A. Belavin[+*1)], *B. Eremin*[*×∘1)], *S. Parkhomenko*[+×1)]

[+]*Landau Institute for Theoretical Physics, 142432 Chernogolovka, Russia*

[*]*Kharkevich Institute for Information Transmission Problems, 127051 Moscow, Russia*

[×]*Moscow Institute of Physics and Technology, 115184 Dolgoprudny, Russia*

[∘]*Skolkovo Institute of Science and Technology, 121205 Moscow, Russia*

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Ten-dimensional Superstring theory unifies the Standard Model of the strong, electromagnetic, and weak interactions with quantum gravity. Starting with 10-dimensional superstring theory, we can get a 4-dimensional theory with spacetime supersymmetry following the Kaluza–Klein idea by compactifying six of the ten dimensions.

For phenomenological reasons we need to do this while maintaining $N = 1$ Supersymmetry of 4-dimensional spacetime. To achieve this, as Candelas, Horowitz, Strominger and Witten have shown, we must compactify six of the ten dimensions of to the so called Calabi–Yau (CY) manifolds.

Another equivalent approach developed by D. Gepner is the compactification of 6 dimensions onto some $N = 2$ Superconformal Field theory with the central charge $c = 9$. Each of these two equivalent approaches has its own merits. Say, using exactly solvability of the Minimal models of $N = 2$ Superconformal Field Theory, it is possible to obtain the explicit solution of the considered models.

In a series of papers [1, 2], we proposed a new method for calculating special geometry on the moduli space of the complex structures of CY manifolds, defined as a hypersurfaces in a weighted projective space. This approach is based on the connection between the cohomology of CY manifolds and their periods, specified by oscillatory integrals, with supersymmetric Landau–Ginzburg models. Knowing the periods allows us to calculate the main components of the special geometry of the moduli space. Using this calculation, an explicit expression was found for the Kähler potential of the metric of special geometry for an arbitrary surface of Fermat type.

Previously, Jockers et al. proposed a conjecture (JKLMR conjecture), which establishes a connection between Kähler potentials in the moduli space of Kähler structures of CY manifolds and the exact partition function of $N = (2, 2)$ supersymmetric gauge linear sigma models (GLSM) on a two-dimensional sphere. We tested the JKLMR conjecture for several simple examples. To do this, in [3, 4] we used our results for the explicit expression for the Kähler potential together with mirror symmetry.

We also tested the mirror version of the JKLMR conjecture using the example of the CY manifold X, which does not belong to the Fermat class [5]. Moreover, for the general class of Berglund–Hübsch families, a connection was found between GLSM for Y and the CY family X [6].

In the work [7], devoted to the study of CY manifolds of the Berglund–Hübsch type, we discovered the phenomenon of coincidence of the Kähler potential on the moduli space of complex structures of two different CY families. Two cases of such coincidences were considered. Each pair has two families of Calabi–Yau manifolds with matching Hodge numbers. For both families in each pair, using our method proposed in [1], the Kähler potentials were found on the moduli space of complex structures. After this, the fact that the calculated potentials coincide in each pair was checked. To do this for both families in each pair a so-called fan that defines this CY family was constructed.

In the works [8, 9] we studied the Mirror symmetry for various examples of CY manifolds. We checked the

1)e-mail: belavin@itp.ac.ru; Eremin.ba@phystech.edu; spark@itp.ac.ru

equivalence of the results of constructing a mirror partners using the Berglund–Hübsch–Krawitz constructions and Batyrev constructions.

The phenomenon of multiple Calabi–Yau mirrors, arising in the BHK mirror construction, was found and investigated in [10]. It has been shown that for any pair of CY orbifolds that are BHK mirrors of a pair of loop-chain CY 3-folds, appeared the same weighted projective space, the periods of the holomorphic nonvanishing form are the same.

Based on the results of the work [10] we considered [11] the problem of fulfilling the JKLMR hypothesis mentioned above. The problem was studied for a specific class of CY manifolds that do not belong to the Fermat type class. Namely, the JKLMR conjecture was verified to be true when a CY $X(1)$ of this type has a mirror partner $Y(1)$ in a weighted projective space that also admits a CY of Fermat type $Y(2)$.

In the framework of Gepner a construction of special models of $N = 2$ superconformal field theory was proposed, namely the construction of the tensor product of two-dimensional $N = 2$ superconformal minimal field theories [12, 13]. It is shown that the construction which ensures the mutual locality also ensures the modular invariance. In addition, the exact connection was demonstrated between the CY orbifolds defined in the weighted projective space and the orbifolds of the product of $N = 2$ minimal superconformal models.

The review concludes with the work [14] on the cases where CY orbifolds of different Berglund–Hubsch types arise as hypersurfaces in the same weighted projective space. There established the explicit connection between such CY threefolds. Also, a simple proof of the birationality of their mirror partners appearing in two different weighted projective spaces is also given. The latter fact explains the coincidence of the periods of these mirrors, proven in our previous work.

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